Advanced Topics in Geometry A1 (MTH.B405)

An application

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Exercise 6-1

$0 < \theta < \pi$ Gauss eq.

Problem (Ex. 6-1)

Let $\theta \colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming θ satisfies $\boxed{\theta_{uv} = \sin\theta}$, prove that there exists a surface $p \colon U \to \mathbb{R}^3$ whose first and second fundamental forms are

- $ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2$. $\left(\det \hat{I} = 1 - \cos^2\theta = \sin^2\theta\right)$
- $II = 2\sin\theta\, du\, dv$.

$$\hat{I} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} \qquad \hat{II} = \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix}$$

$\Rightarrow$ the Gaussian curvature $K = \dfrac{\det \hat{II}}{\det \hat{I}} = -1$

Q and A

Q: We need the first fundamental form to be positive definite, in particular invertible. If $\theta \in (0, \pi)$ then there might exist $(u, v) \in U$ such that $\underline{\cos^2\theta(u, v) = 1}$. If we replace the first fundamental form by $ds^2 = 1\,du^2 + \cos\theta\,du\,dv + 1\,dv^2$, then ds^2 is positive definite and the Codazzi equations are satisfied.

$\boxed{|\cos\theta| < 1}$

Exercise 6-1

satisfy Codazzi equations automatically.

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv.$$

- Codazzi equations are satisfied (cf. Ex. 5-1)

Problem (Ex. 5-1)

Assume $L = N = 0$, *that is,* $II = 2M\, du\, dv = 2h_{12}\, du^1\, du^2$, *Prove that, if the Gaussian curvature* K *is negative constant,*

$$E_v = G_u = 0, \qquad \textit{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$

$E = 1 \quad G = 1$

Exercise 6-1

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv.$$

- Gauss-Weingarten equations (cf. Ex. 4-1)

$$\mathcal{F}_u = \mathcal{F}\Omega, \qquad \mathcal{F}_v = \mathcal{F}\Lambda,$$

$$\Omega = \begin{pmatrix} \theta_u \cot\theta & 0 & \cot\theta \\ -\theta_u \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 0 & -\theta_v \csc\theta & -\csc\theta \\ 0 & \theta_v \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}$$

- Integrablity Conditions: $\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega = O$

$\rightsquigarrow \qquad \Leftrightarrow \theta_{uv} = \sin\theta$

Gauss.

Q and A

Q: In problem 6-1, it is shown that $\theta_{uv} = \sin\theta$ is required from the Gauss equation, but can this seemingly neat condition $\theta_{uv} = \sin\theta$ be found in other ways? Or can it be found only by calculating the Gauss equation?

$\Delta\sigma = \sigma_{uu} + \sigma_{vv}$

Problem (Ex. 6-2)

Let $\sigma\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming σ satisfies $\Delta\sigma = -\frac{1}{2}\sinh 2\sigma$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big).$$

isothermal parameter

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad II = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big).$$

$L = \frac{1}{2}(e^{2\sigma}+1)$, $N = \frac{1}{2}(e^{2\sigma}-1)$

$M = 0$

- Codazzi equations are satisfied (cf. Ex. 5-2)

$$\frac{\partial q}{\partial \bar z}^{\,0} = e^{2\sigma}\frac{\partial H}{\partial z}^{\,0}$$

$$q = \frac{L-N}{2} - iM = \frac{1}{2}$$

$$H = \frac{1}{2}e^{-2\sigma}(L+N) = \frac{1}{2}$$

$$z = u + iv$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

Problem 6-2

Problem (Ex. 5-2)

Assume $F = 0$ *and* $E = G = e^{2\sigma}$, *where* σ *is a function Let* $z = u + iv$ $(i = \sqrt{-1})$ *and define a complex-valued function* q *by*

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma}\frac{\partial H}{\partial z},$$

where H *is the mean curvature, and*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Problem 6-2

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad II = \frac{1}{2}\big(\underbrace{(e^{2\sigma} + 1)}_{L}du^2 + \underbrace{(e^{2\sigma} - 1)}_{N}dv^2\big).$$

$M = 0$

- Gauss-Weingraten equations (cf. Ex. 4-1)

$\mathcal{F}_u = \mathcal{F}\Omega$

$\mathcal{F}_v = \mathcal{F}\Lambda$

$$\Omega = \begin{pmatrix} 0 & -\sigma_v & -e^{-\sigma}L \\ \sigma_v & 0 & -e^{-\sigma}M \\ e^{-\sigma}L & e^{-\sigma}M & 0 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 0 & -\sigma_u & -e^{-\sigma}M \\ \sigma_u & 0 & -e^{-\sigma}N \\ e^{-\sigma}M & e^{-\sigma}N & 0 \end{pmatrix}$$

Problem 6-2

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad II = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big).$$

- Gauss-Weingraten equations (cf. Ex. 4-1)

$$\Omega = \begin{pmatrix} 0 & -\sigma_v & -e^{-\sigma}L \\ \sigma_v & 0 & -e^{-\sigma}M \\ e^{-\sigma}L & e^{-\sigma}M & 0 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 0 & -\sigma_u & -e^{-\sigma}M \\ \sigma_u & 0 & -e^{-\sigma}N \\ e^{-\sigma}M & e^{-\sigma}N & 0 \end{pmatrix}$$

Compute integrability : $\Leftrightarrow \Delta\sigma = -\frac{1}{2}\sinh 2\sigma.$