Advanced Topics in Geometry A1 (MTH.B405)

An application

constant mean curvature surfaces

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Mean Curvature

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad II = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

$\Rightarrow$

$$H = \frac{1}{2}\operatorname{tr} A = \frac{EN - 2FM + GL}{2(EG - F^2)}; \qquad A := \widehat{I}^{-1}\,\widehat{II}.$$

mean curvature
平均曲率
heikinkyokuritsu

(die Krümmung → K)

Mean Curvature

Q (May 23): Since the principal curvatures are the curvature of the most curved direction and the Gaussian curvature is the product of the two of them, I think I can get roughly a shape of surface from the sign of Gaussian curvature. On the other hand, I am not sure what the mean curvature geometrically. Is it something that can be intuitively understood?

Area and mean curvature

Definition (Area of surfaces)

closure

$p\colon U \to \mathbb{R}^3$: a surface; $V \subset U$ ($\overline{V} \subset U$ is compact)

area element

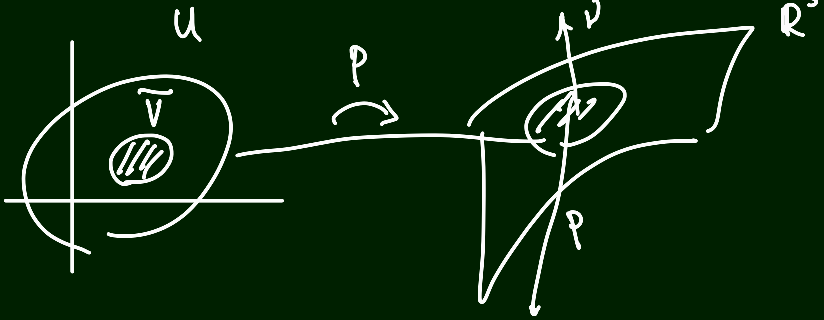
area

$$\mathcal{A}_p(\overline{V}) := \iint_{\overline{V}} da, \quad da := \sqrt{\det \widehat{I}}\, du\, dv = \sqrt{EG - F^2}\, du\, dv.$$

Proposition (7.3)

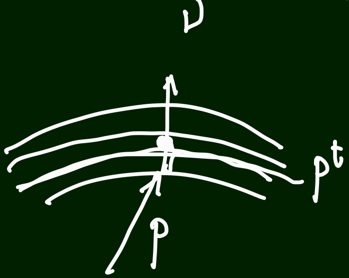
$$\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t \iint_{\overline{V}} H\, da + o(t) \qquad (t \to 0),$$

where $\underline{p^t := p + t\nu}$ (*parallel surfaces*)

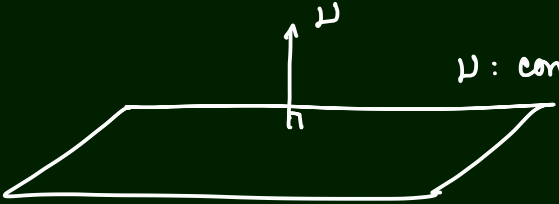


$$P^t := P + t\nu$$

parallel surf



$\mathbb{R}^3$



ν : constant vector

$$d\nu = 0 \Rightarrow \hat{\mathrm{I\!I}} = 0$$
$$\Rightarrow A = 0$$
$$\Rightarrow H = 0$$

Minimal Surfaces

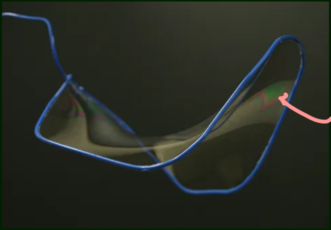
the Euler-Lagrange for area functional is $H=0$

Fact

If a surface $p \in \mathcal{S}_C$ has the least area among all surfaces in $\mathcal{S}_C$, then the mean curvature of p identically vanishes.

Definition

A minimal surface is a surface whose mean curvature vanishes identically.

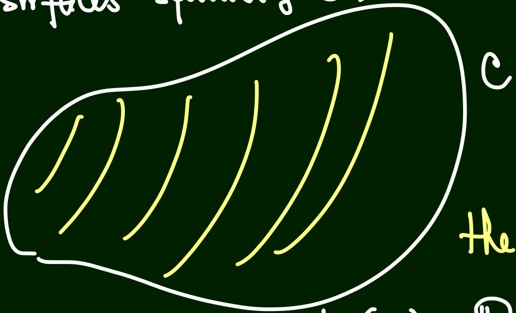


soap film

https://math.hmc.edu/jacobsen/demolab/soap-film-2/

C : a simple closed curve in $\mathbb{R}^3$

$\mathcal{S}_C := \{$ surfaces spanning $C \}$



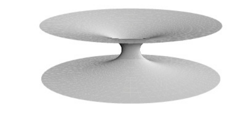
the area functional

$$\mathcal{A} : \mathcal{S}_C \ni p \longmapsto \underbrace{A_p(D)}_{\geq 0} \in \mathbb{R}$$

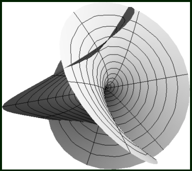
$\exists \inf$

$\sup = \infty$

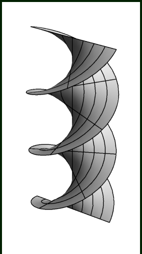
Minimal Surfaces



catenoid



Enneper's



helicoid

Broad area in differential geometry.

Minimal Surfaces

Olympiastadion München (1971); Frei Otto

Lemma (7.7)

Let $S \subset \mathbb{R}^3$ be a surface Assume for all P and $\mathrm{Q} \in S$, there exists an orientation preserving congruence F of $\mathbb{R}^3$ satisfying $F(S) = S$ and $F(\mathrm{P}) = \mathrm{Q}$. Then the mean curvature of S is constant.

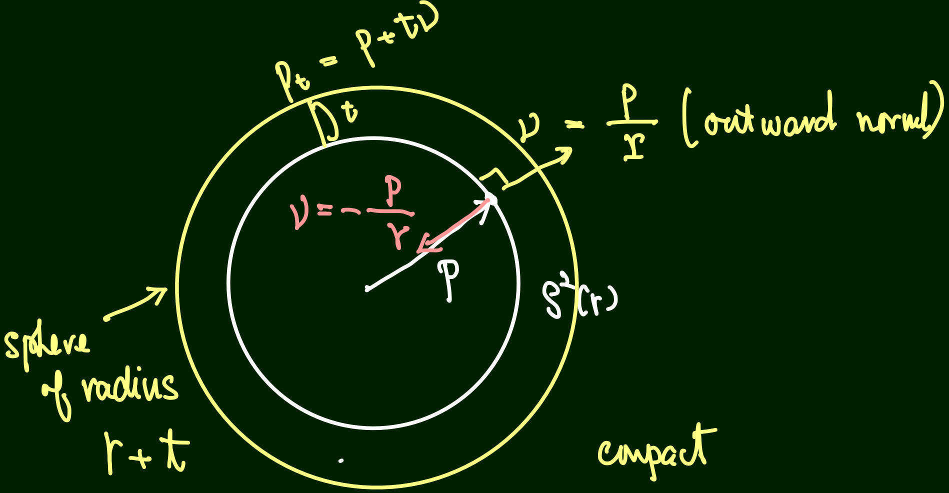
$$S = S^2(r) := \{(x, y, z)\,;\, x^2 + y^2 + z^2 = r^2\} \Rightarrow H = \frac{\pm 1}{r}$$

$$\boxed{\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t \iint_{\overline{V}} H\, da + o(t)}$$

H : const

H = $-\frac{1}{r}$ (outward normal)

= $\frac{1}{r}$ (inward normal)

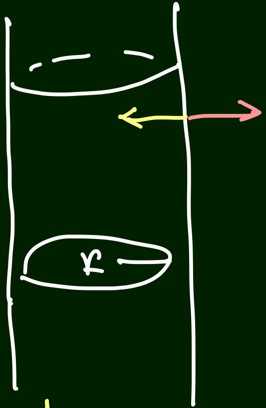


p_t = p + tν
t
ν = p/r (outward normal)
ν = −p/r
p
S²(r)
sphere of radius r + t
compact

Example—The (circular) cylinder

$$S = \{(x, y, z)\,;\, x^2 + y^2 = r^2\} \Rightarrow H = \frac{\pm 1}{2r}$$

$$\boxed{\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t \iint_{\overline{V}} H \, da + o(t)}$$



non compact

(closed)

Constant Mean Curvature (CMC) Surfaces

Fact

When the volume of the enclosed domain is fixed, the closed surface with the least area is of (non-zero) constant mean curvature.

$\underline{H = \text{const}}$

soap bubble

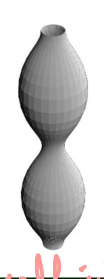
https://en.wikipedia.org/wiki/Soap_bubble

CMC surfaces

compact ?

Q

Are there any other constant mean curvature surfaces than the "trivial" examples above?



surface of revolution (non compact)

no self intersections

unduloid

self intersections.

nodoid

Delaunay's surfaces

Fact (A. D. Alexandrov 1958)

The only closed surfaces of constant mean curvature without self-intersections are the round spheres.

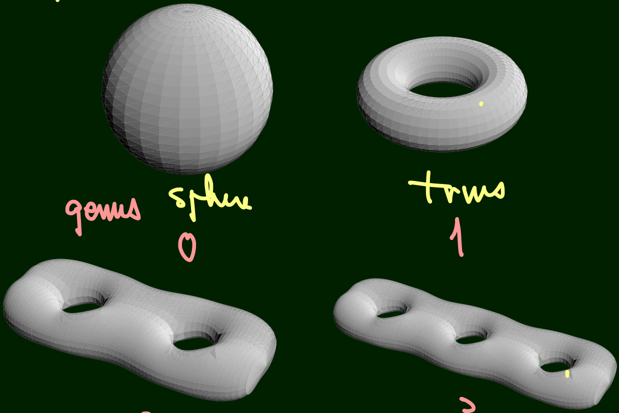
compact

$\nexists$ self-intersection $\Rightarrow$ no example other than sphere

- symmetrization
- maximum principle of elliptic PDE.

Classification of Closed Surfaces

topological types of "oriented" closed surfaces:



<u>Thm</u> H. Hopf. (1953)

The only closed constant mean curvature surface of <u>genus 0</u> is the round sphere.

- Liouville's theorem in complex analysis
- Codazzi eq.

<u>Problem</u> (Hopf's problem)

$\exists ?$ closed constant mean curvature surface other than round spheres?

CMC tori

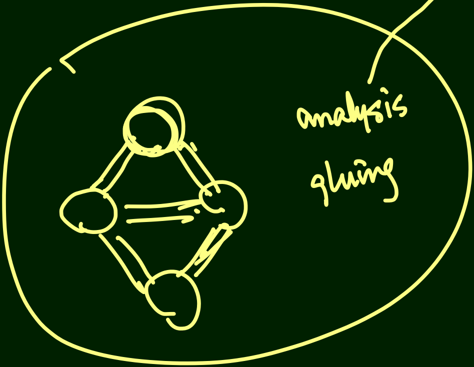
constant mean curvature

Theorem (H. Wente, 1986)

There exists a CMC torus immersed in $\mathbb{R}^3$.

Gauss & Codazzi

Higher genus examples: N. Kapouleas, 1986.



Wente's construction—How to describe tori

period of torus

Definition

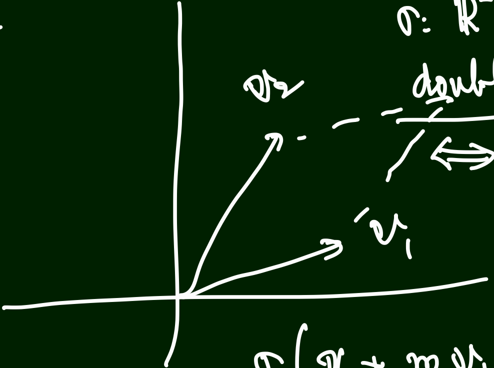
A function f defined on $\mathbb{R}^2$ is said to be doubly periodic if there exists a pair $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of linearly independent vectors in $\mathbb{R}^2$ such that

$$f(\boldsymbol{x}+\boldsymbol{v}_1) = f(\boldsymbol{x}+\boldsymbol{v}_2) = f(\boldsymbol{x}) \tag{1}$$

holds for any $\boldsymbol{x} \in \mathbb{R}^2$. The basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is called the period of f.

- A doubly periodic function f is considered as a function on a torus $\mathbb{R}^2/(\mathbb{Z}\boldsymbol{v}_1 \oplus \mathbb{Z}\boldsymbol{v}_2)$.
- Goal: to construct doubly periodic $p\colon \mathbb{R}^2 \to \mathbb{R}^3$.

$\mathbb{R}^2$



$\sigma: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is doubly periodic

$\Longleftrightarrow$ $\sigma(x + v_1) = \sigma(x + v_2) = \sigma(x)$

$$\sigma(x + m v_1 + n v_2) = \sigma(x)$$

$m, n \in \mathbb{Z}$

$\mathbb{R}^2 / (\mathbb{Z} v_1 \oplus \mathbb{Z} v_2) = M$ compact genus 1

parallel translation in $\mathbb{R}^2$

Wente's construction—Fundamental Theorem

Proposition

Let $\sigma\colon \mathbb{R}^2 \to \mathbb{R}$ be a doubly periodic function with period $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. If σ satisfies

$$\boxed{\Delta\sigma = \sigma_{uu} + \sigma_{vv} = -\frac{1}{2}\sinh 2\sigma,}$$

there exists a parametrized surface $\boldsymbol{p}\colon \mathbb{R}^2 \to \mathbb{R}^3$ with

$H = \frac{1}{2}$

Prob 6-2

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad II = \frac{1}{2}\left((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\right),$$

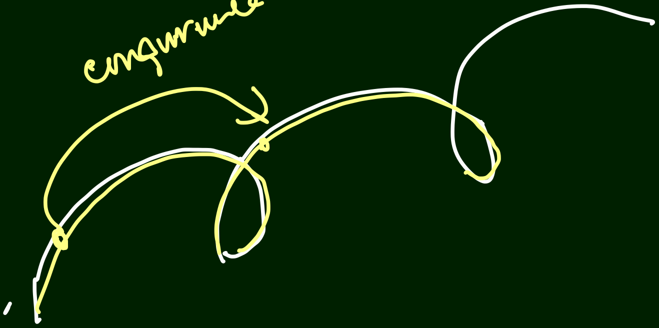
whose mean curvature is identically $1/2$. Moreover, there exist matrices $R_i \in \mathrm{SO}(3)$ and vectors $\boldsymbol{a}_i \in \mathbb{R}^3$ $(i = 1, 2)$ such that

$p(x)$ & $p(x+v_i)$ have common fundamental forms

$$\boldsymbol{p}(\boldsymbol{x} + \boldsymbol{v}_i) = R_i \boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{a}_i \qquad (i = 1, 2).$$

in general $a_i \neq 0$, $R_i \neq \mathrm{id}$

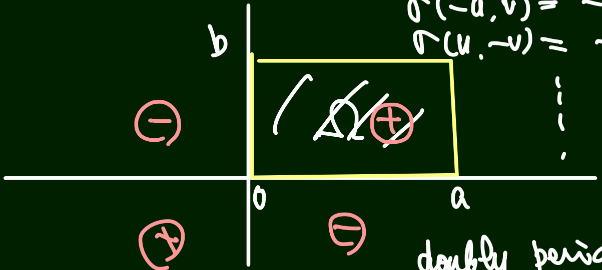
смещение



Wente's construction—How to Solve Gauss eq.

$\Omega := (0,a) \times (0,b)$

$$\Delta\sigma = -\frac{1}{2}\sinh 2\sigma \quad \text{on } \Omega, \qquad \sigma = 0 \quad \text{on } \partial\Omega, \qquad \sigma > 0 \quad \text{on } \Omega^o,$$



$\sigma(-u,v) = -\sigma(u,v)$
$\sigma(u,-v) = -\sigma(u,v)$
$\vdots$

doubly periodic func σ on $\mathbb{R}^2$

periods: $(2a,0)$ $(0,2b)$

$$\boxed{\boldsymbol{p}(\boldsymbol{x}+\boldsymbol{v}_i) = R_i\boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{a}_i}$$

$\theta \neq 0$

$R_2 = \mathrm{id}$, $\boldsymbol{a}_i = \boldsymbol{0}$ $(i = 1, 2)$, and

v_2: periodic

$$R_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\theta = \theta(a, b)$.

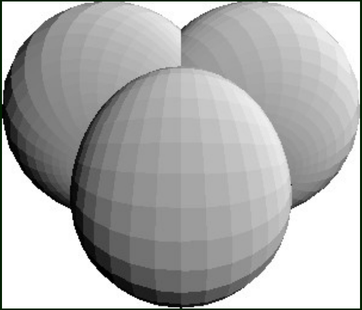
$\boxed{\text{non const}}$

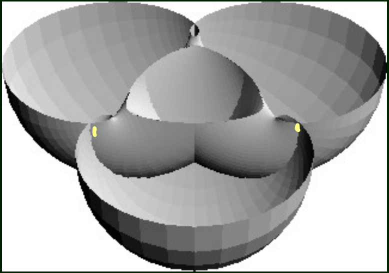
$\exists (a,b)$ s.t. $\frac{\theta}{2\pi} \in \mathbb{Q}$

$\Rightarrow R_1^m = \mathrm{id}$ w.r.t. $(2ma, 0)$, $(0, 2b)$

$\Rightarrow P$ is dobly periodic

Wente torus





Jacobi sn functions

After Wente

- U. Abresach (1987)
- R. Walter (1987)
- U. Pinkall and I. Stering (1989)

u-curves & v-curves are planar.

eq. on σ

explicit

σ: given in

(classified)

in terms of

elliptic theta fct; elliptic functions

(integrable systems and differential geometry.)

pseudospherical surfaces; $(K=-1)$

Thank you!