Advanced Topics in Geometry A1 (MTH.B405)

An application

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2025/06/06

Mean Curvature

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

$\Rightarrow$

$$H = \frac{1}{2}\operatorname{tr} A = \frac{EN - 2FM + GL}{2(EG - F^2)}, \qquad A := \widehat{I}^{\,-1}\,\widehat{I\!I}$$

Mean Curvature

Q (May 23): Since the principal curvatures are the curvature of the most curved direction and the Gaussian curvature is the product of the two of them, I think I can get roughly a shape of surface from the sign of Gaussian curvature. On the other hand, I am not sure what the mean curvature geometrically. Is it something that can be intuitively understood?

Area and mean curvature

Definition (Area of surfaces)

$p\colon U \to \mathbb{R}^3$: a surface; $V \subset U$ ($\overline{V} \subset U$ is compact)

$$\mathcal{A}_p(\overline{V}) := \iint_{\overline{V}} da, \quad da := \sqrt{\det \widehat{I}}\, du\, dv = \sqrt{EG - F^2}\, du\, dv.$$

Proposition (7.3)

$$\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t \iint_{\overline{V}} H\, da + o(t) \qquad (t \to 0),$$

where $p^t := p + t\nu$ (*parallel surfaces*)

Example—The Plane

Minimal Surfaces

Fact

If a surface $p \in \mathcal{S}_C$ has the least area among all surfaces in $\mathcal{S}_C$, then the mean curvature of p identically vanishes.

Definition

A <u>minimal surface</u> is a surface whose mean curvature vanishes identically.

Example—The Round sphere

Lemma (7.7)

Let $S \subset \mathbb{R}^3$ be a surface Assume for all P and $\mathrm{Q} \in S$, there exists an orientation preserving congruence F of $\mathbb{R}^3$ satisfying $F(S) = S$ and $F(\mathrm{P}) = \mathrm{Q}$. Then the mean curvature of S is constant.

$$S = S^2(r) := \{(x, y, z)\,;\, x^2 + y^2 + z^2 = r^2\} \Rightarrow H = \frac{\pm 1}{r}$$

$$\boxed{\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t \iint_{\overline{V}} H\, da + o(t)}$$

Example—The (circular) cylinder

$$S = \{(x, y, z)\,;\, x^2 + y^2 = r^2\} \Rightarrow H = \frac{\pm 1}{r}$$

$$\boxed{\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t \iint_{\overline{V}} H\, da + o(t)}$$

Constant Mean Curvature (CMC) Surfaces

Fact

When the volume of the enclosed domain is fixed, the closed surface with the least area is of (non-zero) constant mean curvature.

CMC surfaces

Q

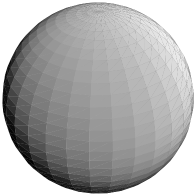
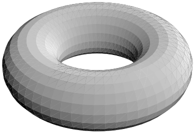
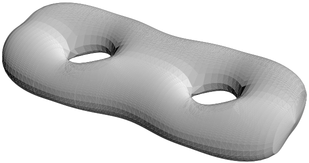
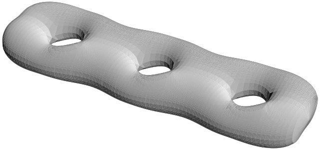
Are there any other constant mean curvature surfaces than the “trivial” examples above?

Alexandrov's theorem

Fact (A. D. Alexandrov 1958)

The only closed surfaces of constant mean curvature without self-intersections are the round spheres.

Classification of Closed Surfaces

CMC tori

Theorem (H. Wente, 1986)

There exists a CMC torus immersed in $\mathbb{R}^3$.

Higher genus examples: N. Kapouleas, 1986.

Wente's construction—How to describe tori

Definition

A function f defined on $\mathbb{R}^2$ is said to be doubly periodic if there exists a pair $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of linearly independent vectors in $\mathbb{R}^2$ such that

$$f(\boldsymbol{x} + \boldsymbol{v}_1) = f(\boldsymbol{x} + \boldsymbol{v}_2) = f(\boldsymbol{x}) \tag{1}$$

holds for any $\boldsymbol{x} \in \mathbb{R}^2$. The basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is called the period of f.

- A doubly periodic function f is considered as a function on a torus $\mathbb{R}^2/(\mathbb{Z}\boldsymbol{v}_1 \oplus \mathbb{Z}\boldsymbol{v}_{\mathbf{2}})$.
- Goal: to construct doubly periodic $p\colon \mathbb{R}^2 \to \mathbb{R}^3$.

Wente's construction—Fundamental Theorem

Proposition

Let $\sigma\colon \mathbb{R}^2 \to \mathbb{R}$ *be a doubly periodic function with period* $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$*. If* σ *satisfies*

$$\Delta\sigma = \sigma_{uu} + \sigma_{vv} = -\frac{1}{2}\sinh 2\sigma,$$

there exists a parametrized surface $\boldsymbol{p}\colon \mathbb{R}^2 \to \mathbb{R}^3$ *with*

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad I\!I = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big),$$

whose mean curvature is identically $1/2$*. Moreover, there exist matrices* $R_i \in \mathrm{SO}(3)$ *and vectors* $\boldsymbol{a}_i \in \mathbb{R}^3$ $(i = 1, 2)$ *such that*

$$\boldsymbol{p}(\boldsymbol{x} + \boldsymbol{v}_i) = R_i \boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{a}_i \qquad (i = 1, 2).$$

Wente's construction—How to Solve Gauss eq.

$\Omega := (0, a) \times (0, b)$

$$\Delta\sigma = -\frac{1}{2}\sinh 2\sigma \quad \text{on } \Omega, \qquad \sigma = 0 \quad \text{on } \partial\Omega, \qquad \sigma > 0 \quad \text{on } \Omega^o,$$

Wente's construction—How to Control Periods

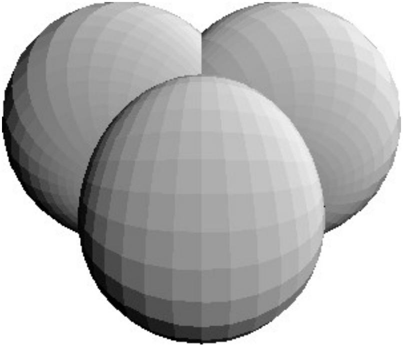
$$\boxed{\boldsymbol{p}(\boldsymbol{x}+\boldsymbol{v}_i) = R_i \boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{a}_i}$$

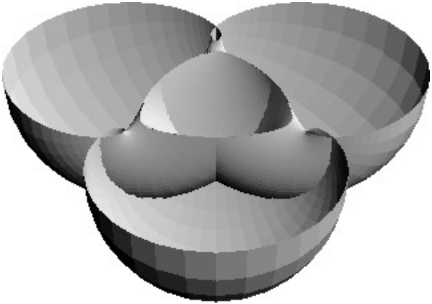
$R_2 = \mathrm{id}$, $\boldsymbol{a}_i = \boldsymbol{0}$ $(i = 1, 2)$, and

$$R_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\theta = \theta(a, b)$.

Wente torus





After Wente

- U. Abresach (1987)
- R. Walter (1987)
- U. Pinkall and I. Stering (1989)

Thank you!