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Kotaro Yamada
kotaro@math.sci.isct.ac.jp

# Info. Sheet 7; Advanced Topics in Geometry A1 (MTH.B405)

## Informations

- Today is the final class of MTH.B405. Thank you for attending and cooperating the course.
- Please fill the form "Course Survey" in LMS.

## Corrections

- 20250523-C-handout.pdf, page 8, the second line of definition of $R_{jk}$: $\Gamma^s_{ks} \Rightarrow \Gamma^s_{k2}$.
- Lecture Note, page 31, eq. (6.3):

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{kj,i} + g_{ik,j} - g_{ij,k}) \qquad \Rightarrow \qquad \Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{lj,i} + g_{il,j} - g_{ij,l})$$

- Lecture Note, page 32, proof of Lemma 6.2: $g_{ij} \Rightarrow \gamma_{ij}$.
- Lecture Note page 32 (and page 7 of Handout/Blackboard C), Problem 6-2:

$$\Delta\sigma = -\frac{1}{2}\sinh\sigma \qquad \Rightarrow \qquad \Delta\sigma = -\frac{1}{2}\sinh 2\sigma$$

- Lecture Note, page 33, line 1: it hold ⇒ it holds

## Students' comments

- このあたりは計算量が多くて苦手なので，早く 2Q の非ユークリッド幾何に入ってほしいという気持ちです．
  I'm not very good at this field because of amount of calculations, so I'm hoping to get into 2Q non-Euclidean geometry as soon as possible.

  **Lecturer's comment** A lot of calculations are included in 2Q's lectures.

## Q and A

**Q 1:** （問題 6-1 について）今回のコダッチ方程式は自動的に成立して，ガウス方程式を見たすには $\theta_{uv} = \sin\theta$ が必要になりました．一方，ガウス方程式の方が成り立ちにくい，もしくはそのような例を多く扱うことはありますか？
(For Problem 6-1) This time the Codazzi equations hold automatically and we need $\theta_{uv} = \sin\theta$ to see the Gaussian equation. Is it usual to deal with Gaussian equations that are harder to solve than Codazzi?

**A:** Depending the problems. This time, and in the lectures in 2Q, we first solve Codazzi equations, and Gauss then.

**Q 2:** What fails if $U$is not simply connected?

**A:** The solution might be multi-valued, like Problem 1-2.

**Q 3:** Do we have any example when $(g_{ij}, h_{ij})$ does not satisfy Gauss equation or Codazzi equation, and show that such surface does NOT exist?

**A:** If the fundamental forms do not satisfy Gauss/Codazzi, the corresponding surface does not exist (automatically) because the Gauss and Weingarten equations do not satisfy integrability conditions. For example, $ds^2 = dx^2 + dy^2$ be $I\!I = dx^2 + dy^2$ cannot be realized by any surface in $\mathbb{R}^3$.

**Q 4:** Thm 6.1 は回転と平行移動を除いて一意ということですよね?
Does the uniquness in Theorem 6.1 mean unique up to rotations and translations?

**A:** Yes.

**Q 5:** 問題 6-1 において Gauss equation から $\theta_{uv} = \sin\theta$ が必要であることが示されますが，この $\theta_{uv} = \sin\theta$ という一見きれいな条件は，他の方法でも分かるのでしょうか？ それとも Gauss equation を計算して初めてわかるのでしょうか.
In problem 6-1, it is shown that $\theta_{uv} = \sin\theta$ is required from the Gauss equation, but can this seemingly neat condition $\theta_{uv} = \sin\theta$ be found in other ways? Or can it be found only by calculating the Gauss equation?

**A:** The Gauss-Weingarten equations for these fundamental forms are simple (cf. Problem 4-2). The Gauss equation is the integrability condition of these.

**Q 6:** What happens if two symmetric matrices $\widehat{I}$ and $\widehat{II}$ with components that are real-valued $C^\infty$-functions on $U$ satisfy only Gauss equation? Or only Codazzi's? Can anything be said about the existence of regular surfaces $p$ with fitting fundamental forms in such cases?

**A:** No. The corresponding surface cannot exist because the compatibility condition of the Gauss frame fails.

**Q 7:** We need the first fundamental form to be positive definite, in particular invertible. If $\theta \in (0, \pi)$ then there might exist $(u, v) \in U$ such that $\cos^2\theta(u, v) = 1$. If we replace the first fundamental form by $ds^2 = 1\,du^2 + \cos\theta\,du\,dv + 1\,dv^2$, then $ds^2$ is positive definite and the Codazzi equations are satisfied.

**A:** When $\theta \in (0, \pi)$, $|\cos\theta| < 1$, never reaching 1.