1 Overview

Euclidean space

In this lecture, we denote by $\mathbb{R}^n$ the n-dimensional *Euclidean space* with canonical *inner product* $\langle\ ,\ \rangle$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T \boldsymbol{y} = x_1 y_1 + \cdots + x_n y_n \qquad \text{for} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n, \tag{1.1}$$

here, we regard an element of $\mathbb{R}^n$ as a column vector, and $(*)^T$ denotes the matrix transposition. Set[1]

$$\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}, \qquad d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{y} - \boldsymbol{x}\| \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n) \tag{1.2}$$

which is called the *norm* of $\boldsymbol{x}$, and the *distance* of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.

A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *isometry* if

$$d\bigl(f(\boldsymbol{x}), f(\boldsymbol{y})\bigr) = d(\boldsymbol{x}, \boldsymbol{y}) \tag{1.3}$$

holds for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^n$.

Definition 1.1. An $n \times n$ real matrix R is said to be an *orthogonal matrix* if $R^T R = \mathrm{id}$ holds, where id is the $n \times n$ *identity matrix.*

The determinant of an orthogonal matrix R is 1 or -1. We denote by $\mathrm{O}(n)$ the set of $n \times n$ orthogonal matrices, and

$$\mathrm{SO}(n) := \{R \in \mathrm{O}(n)\,;\, \det R = 1\}. \tag{1.4}$$

Fact 1.2. *A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is isometry if and only if it is written in the form*

$$f(\boldsymbol{x}) = R\boldsymbol{x} + \boldsymbol{a} \qquad \bigl(R \in \mathrm{O}(n), \boldsymbol{a} \in \mathbb{R}^n\bigr). \tag{1.5}$$

If R in (1.5) is a member of $\mathrm{SO}(n)$, f is said to be *orientation preserving.*

The Fundamental Theorem for surface Theory

Our object in this lecture is *surfaces* in Euclidean 3-space. The simplest question is:

Question 1.3. *What quantity determines a shape of surface?*

It is necessary for mathematical formulation of this question to express the surface. Among several ways to explain surfaces, we regard a surface as a *parametrization*, that is, a map [2]

$$\boldsymbol{f}\colon U \ni (u, v) \mapsto \boldsymbol{f}(u, v) \in \mathbb{R}^3,$$

where U is a *domain* [3]of $\mathbb{R}^2$.

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[1] "$A := B$" means that "A is defined by B".

[2] Unless confusion, points in the source domain are represented by row vectors.

[3] A domain is a connected open subset $U \subset \mathbb{R}^n$.

Example 1.4.

- Set $U := (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$\boldsymbol{f} : U \ni (u, v) \mapsto \boldsymbol{f}(u, v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix} \in \mathbb{R}^3$$

  is a parametrization of the unit sphere in $\mathbb{R}^3$. The parameter u (resp. v) represents the longitude (resp. the latitude) of the point of the sphere.

- Set $V := (-\pi, \pi) \times \mathbb{R}$ and

$$\boldsymbol{g} : V \ni (s, t) \mapsto \boldsymbol{g}(s, t) = \begin{pmatrix} \cos s \operatorname{sech} t \\ \sin s \operatorname{sech} t \\ \tanh t \end{pmatrix} \in \mathbb{R}^3.$$

  Then $\boldsymbol{g}$ parametrizes the unit sphere, and the st-plane is regarded as the Mercator's world map.

Then the following "fundamental theorem" is one of the answer:

Theorem (The Fundamental Theorem for surface theory)**.** *Let*

- *$U \subset \mathbb{R}^2$ be a simply connected domain,*
- *I be a positive definite symmetric quadratic form on U*
- *II be a symmetric quadratic form on U.*

Assume I and II satisfy the Gauss and Codazzi equations. Then there exists a surface $\boldsymbol{f} \colon U \to \mathbb{R}^3$ whose first and second fundamental forms are I and II, respectively.

Moreover, such an $\boldsymbol{f}$ is unique up to orientation preserving isometry of $\mathbb{R}^3$.

The undefined words in the statement, and mathematical meanings of the theorem will be explained through the lecture, and our goal is to prove this theorem.

Commutativity of partial derivatives

One of the most important fact in undergraduate calculus is the following "commutativity of partial derivatives".

Theorem 1.5. *Let $f \colon U \to \mathbb{R}$ be a function defined on a domain U of $\mathbb{R}^2$ and fix a point $p = (u, v) \in U$. If the second derivative $\partial^2 f/(\partial x \partial y) = f_{yx}$ and $\partial^2 f/(\partial y \partial x) = f_{xy}$ are both defined on U and continuous at p, then*

$$\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)$$

holds.

Proof. Take $(h, k) \in \mathbb{R}^2$ satisfying $(u + th, v + sk) \in U$ for all $t, s \in [0, 1]$. Let

$$g(h, k) := f(u + h, v + k) - f(u, v + k) - f(u + h, v) + f(u, v).$$

Since the partial derivative f_x exists on U, the function of one variable $F_1(t) := g(th, k)$ is differentiable on $0 \leqq t \leqq 1$. Then the mean value theorem implies that there exists $\theta_1 = \theta_1(h, k)$ with $0 < \theta_1 < 1$ such that

$$g(h, k) = F_1(1) = F_1(1) - F_1(0) = F_1'(\theta_1) = \big(f_x(u + \theta_1 h, v + k) - f_x(u + \theta_1 h, v)\big)h = F_2(1)h,$$

where $F_2(s) := f_x(u+\theta_1 h, v+sk) - f_x(u+\theta_1 h, v)$ $(0 \leqq s \leqq 1)$. Since $(f_x)_y$ exists on U, F_2 is differentiable on $0 \leqq s \leqq 1$. So, applying mean value theorem again, there exists $\theta_2 = \theta_2(h,k) \in (0,1)$ such that

$$F_2(1) = F_2'(\theta_2) = f_{xy}(u+\theta_1 h, v+\theta_2 k)k.$$

Summing up, there exists θ_1, $\theta_2 \in (0,1)$ depending on h and k such that

$$g(h,k) = f_{xy}(u+\theta_1 h, v+\theta_2 k)hk. \tag{1.6}$$

On the other hand, changing roles of h and k, we know that there exist φ_1, $\varphi_2 \in (0,1)$ such that

$$g(h,k) = f_{yx}(u+\varphi_1 h, v+\varphi_2 k)hk. \tag{1.7}$$

Then

$$f_{xy}(u+\theta_1 h, v+\theta_2 k) = f_{yx}(u+\varphi_1 h, v+\varphi_2 k)$$

whenever $hk \neq 0$. Here, taking limit $(h,k) \to (0,0)$, we have

$$(u+\theta_1 h, v+\theta_2 k) \to (u,v), \qquad (u+\varphi_1 h, v+\varphi_2 k) \to (u,v)$$

because θ_j, $\varphi_j \in (0,1)$ for $j=1,2$. Thus, by continuity of f_{xy} and f_{yx}, we have $f_{xy}(u,v) = f_{yx}(u,v)$. □

Definition 1.6. A function f defined on a domain $U \subset \mathbb{R}^2$ is said to be

(1) *of class* C^0 if it is continuous on U,

(2) *of class* C^1 if there exists a partial derivative f_x and f_y on U, and both of them are continuous,

(3) *of class* C^r $(r = 2,3,\dots)$ if it is of class C^{r-1} and all of the $(r-1)$-st partial differentials are of class C^1, and

(4) *of class* C^∞ if it is of class C^r for arbitrary non-negative integer r.

Using these terms, we have

Corollary 1.7. *If a function $f\colon U \to \mathbb{R}$ defined on a domain U of $\mathbb{R}^2$ is of class C^2, then $f_{xy} = f_{yx}$ holds on U.*

In this lecture, functions are assumed to be of class C^∞. So partial differentials are always commutative.

Inverse of the commutativity—Poincaré lemma

A *differential* 1*-form*, or a 1*-form* defined on a domain $U \subset \mathbb{R}^2$ is the form

$$\alpha = a(x,y)\,dx + b(x,y)\,dy$$

where a and b are C^∞-functions defined on U. The *total differential*, or simply the *differential*, of C^∞-function f defined as

$$df := f_x\,dx + f_y\,dy$$

is a typical example of differential forms.

A *differential* 2*-form* is a form

$$\omega = c(x,y)\,dx \wedge dy$$

where c is a C^∞-function. The *exterior differential* $d\alpha$

$$d\alpha = d(a\,dx + b\,dy) = (b_x - a_y)\,dx \wedge dy$$

of 1-form $\alpha = a\,dx + b\,dy$ is a typical example.

Lemma 1.8. *Let f be a C^∞-function defined on a domain $U \subset \mathbb{R}^2$. Then $d(df) = 0$ holds.*

Proof. $d(df) = d(f_x\,dx + f_y\,dy) = (f_{yx} - f_{xy})\,dx \wedge dy = 0.$ □

Theorem 1.9 (Poincaré lemma)**.** *Let U be a simply connected domain, and α a differential 1-form defined on U. If $d\alpha = 0$, then there exists a C^∞ function f defined on U such that $df = \alpha$.*

The definition, fundamental properties of simple connectedness will be given in Section 3.

Exercises

1-1 Let $f(x,y) = e^{ax}\cos y$, where a is a constant. Find a function $g(x,y)$ satisfying

$$g_x = -f_y, \qquad g_y = f_x, \qquad g(0,0) = 0.$$

1-2 Let $U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\}$ and consider a 1-form

$$\alpha = a(x,y)\,dx + b(x,y)\,dy := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

on U. Take a point $\mathrm{P} = (r\cos\theta, r\sin\theta) \in U$ ($r > 1$, $0 < \theta < \pi$), and two curves

$$\begin{aligned} c_1(t) &:= \bigl(x_1(t), y_1(t)\bigr) = (\cos t, \sin t) && (0 \leqq t \leqq \theta),\\ c_2(s) &:= \bigl(x_2(s), y_2(s)\bigr) = (s\cos\theta, s\sin\theta) && (1 \leqq s \leqq r), \end{aligned}$$

whose union gives a curve joining $(1,0)$ and P. Compute the line integral

$$\begin{aligned} \int_{c_1\cup c_2} \alpha := &\int_0^\theta \left(a(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + b(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt \right)\\ &+ \int_1^r \left(a(x_2(s), y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds \right). \end{aligned}$$