7 An application—Surfaces of constant mean curvature

7.1 Mean curvature

Let $p\colon U \ni (u,v) \mapsto p(u,v) \in \mathbb{R}^3$ be a regular parametrization of a surface defined on a domain $U \subset \mathbb{R}^2$, and let ν be its unit normal vector field. We write first and second fundamental forms as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where

$$(\widehat{I} :=) \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u \cdot p_u & p_u \cdot p_v \\ p_v \cdot p_u & p_v \cdot p_v \end{pmatrix}, \qquad (\widehat{I\!I} :=) \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} \nu_u \cdot p_u & \nu_u \cdot p_v \\ \nu_v \cdot p_u & \nu_v \cdot p_v \end{pmatrix}.$$

Since the parametrization is regular, the matrix $\widehat{I}$ is positive definite:

$$EG - F^2 > 0, \quad E > 0, \quad G > 0.$$

Then we define the Weingarten matrix A by

$$A = \widehat{I}^{-1}\,\widehat{I\!I}.$$

Definition 7.1. The *mean curvature* of the surface p is defined by

$$H := \frac{1}{2}\operatorname{tr} A = \frac{EN - 2FM + GL}{2(EG - F)^2}.$$

7.2 Area and mean curvature

To explain geometric meanings of mean curvature, we start with the area of surfaces: Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface as in the top of this subsection. Take a subdomain $V \subset U$ such that the closure $\overline{V}$ of V is bounded and contained in U.

Definition 7.2. The *area* of the image $p(\overline{V})$ of the surface is defined as

$$\mathcal{A}_p(\overline{V}) := \iint_{\overline{V}} da, \qquad da := \sqrt{\det \widehat{I}}\,du\,dv = \sqrt{EG - F^2}\,du\,dv.$$

We call da the *area element* of p.

For a real number t, $p^t := p + t\nu$ is called the *parallel surface* of p with distance t.

Proposition 7.3.

$$\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t\iint_{\overline{V}} H\,da + o(t) \qquad (t \to 0).$$

Proof. The coefficient matrix of the first fundamental form of p^t is obtained as

$$\begin{aligned}\widehat{I}^{\,t} := \begin{pmatrix} E^t & F^t \\ F^t & G^t \end{pmatrix} &= \begin{pmatrix} (p_u + t\nu_u)\cdot(p_u + t\nu_u) & (p_u + t\nu_u)\cdot(p_v + t\nu_v) \\ (p_v + t\nu_v)\cdot(p_u + t\nu_u) & (p_v + t\nu_v)\cdot(p_v + t\nu_v) \end{pmatrix} \\ &= \begin{pmatrix} E - 2tL & F - 2tM \\ F - 2tM & G - 2tN \end{pmatrix} + o(t) = \widehat{I} - 2t\,\widehat{I\!I} + o(t).\end{aligned}$$

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Then

$$\begin{aligned}\det \widehat{I}^{\,t} &= (EG - F^2) - 2t(EN - 2FM + GL) + o(t)\\ &= (EG-F^2)\left(1 - 2t\frac{EN-2FM+GL}{EG-F^2} + o(t)\right) = (EG-F^2)\left(1-4tH+o(t)\right).\end{aligned}$$

Hence the area element of p^t is

$$da^t := \sqrt{\det \widehat{I}}\, du\, dv = \sqrt{EG-F^2}\sqrt{1-4tH+o(t)}\, du\, dv = \bigl(1-2tH+o(t)\bigr)\, da$$

Integrating this, we obtain the conclusion. □

Roughly speaking, the mean curvature is the rate of change of the area of a family of parallel surfaces of a surface. The following proposition supports this: We denote by $\overline{D}$ and $S^1 = \partial D$ the unit closed disc $\{(u,v)\,;\,u^2+v^2 \leqq 1\}$ and its boundary, respectively. Let $C \subset \mathbb{R}^3$ be a simple closed curve in $\mathbb{R}^3$ and denote $\mathcal{S}_C$ the set of surfaces $p\colon \overline{D} \to \mathbb{R}^3$ with $p(S^1) = C$.

Fact 7.4. *If a surface $p \in \mathcal{S}_C$ has the least area among all surfaces in $\mathcal{S}_C$, then the mean curvature of p identically vanishes.*

If you are familiar to the variational method, this means that the Euler-Lagrange equation of the area functional $\mathcal{A}\colon \mathcal{S}_C \to \mathbb{R}$ is "$H = 0$". Keeping this fact in mind,

Definition 7.5. A *minimal surface* is a surface whose mean curvature vanishes identically.

On the other hand, the conditional extremal problem for the area functional, we have

Fact 7.6. *When the volume of the enclosed domain is fixed, the closed surface with the least area is of* (*non-zero*) *constant mean curvature.*

7.3 Examples of constant mean curvature surfaces

Since the mean curvature is invariant under congruence of $\mathbb{R}^3$, we have

Lemma 7.7. *Let $S \subset \mathbb{R}^3$ be a surface* (*an image of a parametrized surface*)*. Assume for all* P *and* $\mathrm{Q} \in S$*, there exists an orientation preserving congruence F of the Euclidean* 3*-space satisfying $F(S) = S$ and $F(\mathrm{P}) = \mathrm{Q}$. Then the mean curvature of S is constant.*

Example 7.8 (The plane)**.** A plane $p(u,v) = (u,v,0)$ is a minimal surface. In fact, since the unit normal vector field $\nu = (0,0,1)$ is constant, $I\!I$ vanishes identically.

Example 7.9 (The round sphere)**.** Let $S := S^2(r) \subset \mathbb{R}^3$ be the sphere of radius $r > 0$ centered at the origin. Since the linear action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$ preserves $S^2(r)$ and transitive, the mean curvature of $S^2(r)$ is constant.

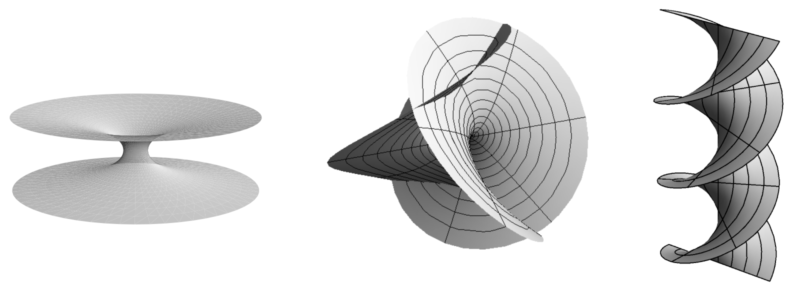
Let us compute the value of the mean curvature: For each point $\boldsymbol{p} \in S^2(r)$, the position vector $\boldsymbol{p}$ is perpendicular to the tangent plane of $S^2(r)$ at $\boldsymbol{p}$. Hence $\boldsymbol{\nu} := (1/r)\boldsymbol{p}$ is the (outward) unit normal vector.

Consider the parallel surface

$$S^t := \left\{\boldsymbol{p} + t\boldsymbol{\nu} = \left(1 + \frac{t}{r}\right)\boldsymbol{p}\,;\,\boldsymbol{p} \in S = S^2(r)\right\},$$

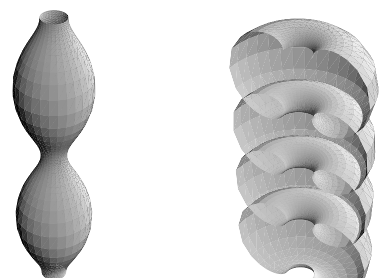
which is the sphere ov radius $(1 + t/r)$. Then

$$\text{Area of } S^t - \text{Area of } S = 4\pi\,(r+t)^2 - 4\pi r^2 = 8\pi r t + O(t^2).$$



catenoid Enneper's helicoid

Figure 1: Minimal surfaces (cf. [UY17])



unduloid nodoid

Figure 2: Delaunay's surfaces (constant mean curvature) (cf. [UY17])

Since the mean curvature H is constant, Proposition 7.3 yields that

$$8\pi r t = -2t \iint_S H\, dA = -2tH(\text{area of } S) = -t \times 8\pi r^2 H.$$

Hence the mean curvature (with respect to the outward unit normal) is $-1/r$.

Similarly, the mean curvature with respect to the inward unit normal is $1/r$.

Example 7.10 (The cylinder)**.** Let S be a circular cylinder of radius r whose axis is the vertical axis of $\mathbb{R}^3$:

$$S = \{\boldsymbol{x} = (x, y, z)\,;\, x^2 + y^2 = r^2\}.$$

Since rotations around the z-axis and vertical translations acts on S transitively, the mean curvature is constant. The same argument as in Example 7.9 works for a finite strip $S' := \{(x, y, z) \in S\,;\, 0 \leqq z \leqq 1\}$, for example, and one can deduce the mean curvature with respect to outward (resp. inward) unit normal is $-1/(2r)$ (resp. $1/2r$).

Question 7.11. *Are there any other constant mean curvature surfaces than the "trivial" examples above?*

7.4 Constant mean curvature surfaces

There are number of examples of constant mean curvature, see Figures 1 and 2.

On the other hand, the following uniqueness theorems are obtained in the middle of 20th century. Here, a *closed surface* means an immersion $p\colon S \to \mathbb{R}^3$ of a compact 2-manifold without boundary.

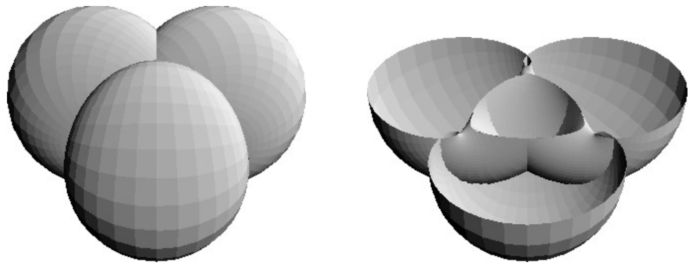


Figure 3: Wente torus

Fact 7.12 (A. D. Alexandrov[Ale58]). *The only closed surfaces of constant mean curvature without self-intersections are the round spheres.*

Fact 7.13 (H. Hopf [Hop53]). *The only closed surfaces of constant mean curvature whose genus zero are the round spheres.*

Then the following problem arises:

Question 7.14 (Hopf's problem). *Are there closed surfaces of constant mean curvature other than the round spheres.*

In 1986, H. Wente constructed constant mean curvature torus [Wen86a] (see Figure 3). Besides, N. Kapouleas also gave examples of constant mean curvature surfaces of genus $\geqq 2$ [Wen86b, BK14]. These two results are obtained by quite different methods. In this lecture, an outline of Wente's construction is introduced as an application of the fundamental theorem for surface theory.

7.5 Wente torus

In this section, we outline the construction of constant mean curvature tori according to Wente [Wen86a].

Definition 7.15. A function f defined on $\mathbb{R}^2$ is said to be *doubly periodic* if there exists a pair $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of linearly independent vectors in $\mathbb{R}^2$ such that

$$f(\boldsymbol{x}+\boldsymbol{v}_1) = f(\boldsymbol{x}+\boldsymbol{v}_2) = f(\boldsymbol{x})y \tag{7.1}$$

holds for any $\boldsymbol{x} \in \mathbb{R}^2$. The basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is called the *period* of f. n

Remark 7.16. If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is doubly periodic with period $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$,

$$f(\boldsymbol{x} + m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2) = f(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^2$$

holds for all $(m_1, m_2) \in \mathbb{Z}^2$. In other words, the function f is invariant under the action of the abelian group

$$\Gamma := \mathbb{Z}\boldsymbol{v}_1 \oplus \mathbb{Z}\boldsymbol{v}_2$$

on $\mathbb{R}^2$ as translations.

Since the quotient space $T := \mathbb{R}^2/\Gamma$ is a smooth 2-manifold diffeomorphic to the torus, the doubly periodic function f is considered as a function on T .

So our goal is

- to construct a doubly periodic constant mean curvature immersion $p\colon \mathbb{R}^2 \to \mathbb{R}^3$.

For the construction, we apply the fundamental theorem for surface theory:

Proposition 7.17. *Let $\sigma\colon \mathbb{R}^2 \to \mathbb{R}$ be a doubly periodic function with period $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. If σ satisfies*

$$\Delta\sigma = \sigma_{uu} + \sigma_{vv} = -\frac{1}{2}\sinh 2\sigma, \tag{7.2}$$

there exists a parametrized surface $\boldsymbol{p}\colon \mathbb{R}^2 \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\bigl((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\bigr), \tag{7.3}$$

whose mean curvature is identically $1/2$. Moreover, there exist matrices $R_i \in \mathrm{SO}(3)$ and vectors $\boldsymbol{a}_i \in \mathbb{R}^3$ $(i = 1, 2)$ such that

$$\boldsymbol{p}(\boldsymbol{x} + \boldsymbol{v}_i) = R_i\boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{a}_i \qquad (i = 1, 2) \tag{7.4}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^2$.

Proof. Exercise 6-2 yields the existence of $\boldsymbol{p}$ with (7.3). Moreover, since $\sigma(\boldsymbol{x}+\boldsymbol{v}_i) = \sigma(\boldsymbol{x})$, $\boldsymbol{p}(\boldsymbol{x}+\boldsymbol{v}_i)$ and $\boldsymbol{p}(\boldsymbol{x})$ have common first and second fundamental forms. Hence the uniqueness of the fundamental theorem implies the existence of R_i and $\boldsymbol{a}_i$ as (7.4). □

In [Wen86a, Section IV], Wente constructed the solutions of (7.2) as follows:

> Let a and b be positive numbers, and set $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$, which is a closed rectangle. First, consider the boundary value problem
>
> $$\Delta\sigma = -\frac{1}{2}\sinh 2\sigma \quad \text{on } \Omega, \qquad \sigma = 0 \quad \text{on } \partial\Omega, \qquad \sigma > 0 \quad \text{on } \Omega^o,$$
>
> where Ω^o is the interior of Ω. Then by reflecting this solution about boundaries, one can extend σ on whole $\mathbb{R}^2$, and the resulting function is doubly periodic with period $\{(2a, 0), (0, 2b)\}$.

Observing the symmetries of σ, one can deduce that $R_2 = \mathrm{id}$, $\boldsymbol{a}_i = \boldsymbol{0}$ $(i = 1, 2)$, and

$$R_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\theta = \theta(a, b)$ is a real number. Moreover, one can show that θ is a non-constant continuous function in (a, b). Hence there exists (a, b) such that $\theta = \theta(a, b) \in 2\pi\mathbb{Q}$. For such (a, b), $R_1^m = \mathrm{id}$ for some integer m. This means that $\boldsymbol{p}$ is $\{(ma, 0), (0, b)\}$-periodic, which yields the example.

After Wente, a lot of results related Wente-type tori are obtained. See, for example, [Abr87, Wal87, PS89].