Introduction

This is a first half of two series of lectures, *Advanced Topics in Geometry A1* and *B1*, in which the fundamental theorem for surface theory and its applications are treated.

Throughout this lecture, object of our interest is “surfaces in Euclidean 3-space”. The goal is to give an comprehensive proof of the fundamental theorem for surface theory ([UY17, Theorem 17.2, see also Appendi B.10]). To accomplish the proof, mathematical tools including the theory of ordinary differential equations and the Frobenius intebrability theorem are expalined.

An aim of the lectures for students is to observe mathematical view around undergraduate calculus and linear algebra.

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1 Overview

Euclidean space

In this lecture, we denote by $\mathbb{R}^n$ the n-dimensional *Euclidean space* with canonical *inner product* $\langle\ ,\ \rangle$:

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^T\boldsymbol{y} = x_1y_1 + \cdots + x_ny_n \qquad \text{for} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n, \tag{1.1}$$

here, we regard an element of $\mathbb{R}^n$ as a column vector, and $(*)^T$ denotes the matrix transposition. Set[1]

$$\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}, \qquad d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{y} - \boldsymbol{x}\| \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n) \tag{1.2}$$

which is called the *norm* of $\boldsymbol{x}$, and the *distance* of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.

A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *isometry* if

$$d\bigl(f(\boldsymbol{x}), f(\boldsymbol{y})\bigr) = d(\boldsymbol{x}, \boldsymbol{y}) \tag{1.3}$$

holds for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^n$.

Definition 1.1. An $n \times n$ real matrix R is said to be an *orthogonal matrix* if $R^TR = \mathrm{id}$ holds, where id is the $n \times n$ *identity matrix*.

The determinant of an orthogonal matrix R is 1 or -1. We denote by $\mathrm{O}(n)$ the set of $n \times n$ orthogonal matrices, and

$$\mathrm{SO}(n) := \{R \in \mathrm{O}(n)\,;\, \det R = 1\}. \tag{1.4}$$

Fact 1.2. *A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is isometry if and only if it is written in the form*

$$f(\boldsymbol{x}) = R\boldsymbol{x} + \boldsymbol{a} \qquad \bigl(R \in \mathrm{O}(n), \boldsymbol{a} \in \mathbb{R}^n\bigr). \tag{1.5}$$

If R in (1.5) is a member of $\mathrm{SO}(n)$, f is said to be *orientation preserving*.

The Fundamental Theorem for surface Theory

Our object in this lecture is *surfaces* in Euclidean 3-space. The simplest question is:

Question 1.3. *What quantity determines a shape of surface?*

It is necessary for mathematical formulation of this question to express the surface. Among several ways to explain surfaces, we regard a surface as a *parametrization*, that is, a map [2]

$$\boldsymbol{f}\colon U \ni (u, v) \mapsto \boldsymbol{f}(u, v) \in \mathbb{R}^3,$$

where U is a *domain* [3]of $\mathbb{R}^2$.

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[1] "$A := B$" means that "A is defined by B".

[2] Unless confusion, points in the source domain are represented by row vectors.

[3] A domain is a connected open subset $U \subset \mathbb{R}^n$.

Example 1.4.

- Set $U := (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$\boldsymbol{f} : U \ni (u, v) \mapsto \boldsymbol{f}(u, v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix} \in \mathbb{R}^3$$

  is a parametrization of the unit sphere in $\mathbb{R}^3$. The parameter u (resp. v) represents the longitude (resp. the latitude) of the point of the sphere.

- Set $V := (-\pi, \pi) \times \mathbb{R}$ and

$$\boldsymbol{g} : V \ni (s, t) \mapsto \boldsymbol{g}(s, t) = \begin{pmatrix} \cos s \operatorname{sech} t \\ \sin s \operatorname{sech} t \\ \tanh t \end{pmatrix} \in \mathbb{R}^3.$$

  Then $\boldsymbol{g}$ parametrizes the unit sphere, and the st-plane is regarded as the Mercator's world map.

Then the following "fundamental theorem" is one of the answer:

Theorem (The Fundamental Theorem for surface theory)**.** *Let*

- *$U \subset \mathbb{R}^2$ be a simply connected domain,*
- *I be a positive definite symmetric quadratic form on U*
- *II be a symmetric quadratic form on U.*

Assume I and II satisfy the Gauss and Codazzi equations. Then there exists a surface $\boldsymbol{f} : U \to \mathbb{R}^3$ whose first and second fundamental forms are I and II, respectively.

Moreover, such an $\boldsymbol{f}$ is unique up to orientation preserving isometry of $\mathbb{R}^3$.

The undefined words in the statement, and mathematical meanings of the theorem will be explained through the lecture, and our goal is to prove this theorem.

Commutativity of partial derivatives

One of the most important fact in undergraduate calculus is the following "commutativity of partial derivatives".

Theorem 1.5. *Let $f : U \to \mathbb{R}$ be a function defined on a domain U of $\mathbb{R}^2$ and fix a point $p = (u, v) \in U$. If the second derivative $\partial^2 f/(\partial x \partial y) = f_{yx}$ and $\partial^2 f/(\partial y \partial x) = f_{xy}$ are both defined on U and continuous at p, then*

$$\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)$$

holds.

Proof. Take $(h, k) \in \mathbb{R}^2$ satisfying $(u + th, v + sk) \in U$ for all $t, s \in [0, 1]$. Let

$$g(h, k) := f(u + h, v + k) - f(u, v + k) - f(u + h, v) + f(u, v).$$

Since the partial derivative f_x exists on U, the function of one variable $F_1(t) := g(th, k)$ is differentiable on $0 \leqq t \leqq 1$. Then the mean value theorem implies that there exists $\theta_1 = \theta_1(h, k)$ with $0 < \theta_1 < 1$ such that

$$g(h, k) = F_1(1) = F_1(1) - F_1(0) = F_1'(\theta_1) = \bigl(f_x(u + \theta_1 h, v + k) - f_x(u + \theta_1 h, v)\bigr)h = F_2(1)h,$$

where $F_2(s) := f_x(u+\theta_1 h, v+sk) - f_x(u+\theta_1 h, v)$ $(0 \leqq s \leqq 1)$. Since $(f_x)_y$ exists on U, F_2 is differentiable on $0 \leqq s \leqq 1$. So, applying mean value theorem again, there exists $\theta_2 = \theta_2(h,k) \in (0,1)$ such that

$$F_2(1) = F_2'(\theta_2) = f_{xy}(u+\theta_1 h, v+\theta_2 k)k.$$

Summing up, there exists θ_1, $\theta_2 \in (0,1)$ depending on h and k such that

$$g(h,k) = f_{xy}(u+\theta_1 h, v+\theta_2 k)hk. \tag{1.6}$$

On the other hand, changing roles of h and k, we know that there exist φ_1, $\varphi_2 \in (0,1)$ such that

$$g(h,k) = f_{yx}(u+\varphi_1 h, v+\varphi_2 k)hk. \tag{1.7}$$

Then

$$f_{xy}(u+\theta_1 h, v+\theta_2 k) = f_{yx}(u+\varphi_1 h, v+\varphi_2 k)$$

whenever $hk \neq 0$. Here, taking limit $(h,k) \to (0,0)$, we have

$$(u+\theta_1 h, v+\theta_2 k) \to (u,v), \qquad (u+\varphi_1 h, v+\varphi_2 k) \to (u,v)$$

because θ_j, $\varphi_j \in (0,1)$ for $j = 1,2$. Thus, by continuity of f_{xy} and f_{yx}, we have $f_{xy}(u,v) = f_{yx}(u,v)$. □

Definition 1.6. A function f defined on a domain $U \subset \mathbb{R}^2$ is said to be

(1) *of class* C^0 if it is continuous on U,

(2) *of class* C^1 if there exists a partial derivative f_x and f_y on U, and both of them are continuous,

(3) *of class* C^r $(r = 2,3,\dots)$ if it is of class C^{r-1} and all of the $(r-1)$-st partial differentials are of class C^1, and

(4) *of class* C^∞ if it is of class C^r for arbitrary non-negative integer r.

Using these terms, we have

Corollary 1.7. *If a function $f\colon U \to \mathbb{R}$ defined on a domain U of $\mathbb{R}^2$ is of class C^2, then $f_{xy} = f_{yx}$ holds on U.*

In this lecture, functions are assumed to be of class C^∞. So partial differentials are always commutative.

Inverse of the commutativity—Poincaré lemma

A *differential* 1*-form*, or a 1*-form* defined on a domain $U \subset \mathbb{R}^2$ is the form

$$\alpha = a(x,y)\,dx + b(x,y)\,dy$$

where a and b are C^∞-functions defined on U. The *total differential*, or simply the *differential*, of C^∞-function f defined as

$$df := f_x\,dx + f_y\,dy$$

is a typical example of differential forms.

A *differential* 2*-form* is a form

$$\omega = c(x,y)\,dx \wedge dy$$

where c is a C^∞-function. The *exterior differential* $d\alpha$

$$d\alpha = d(a\,dx + b\,dy) = (b_x - a_y)\,dx \wedge dy$$

of 1-form $\alpha = a\,dx + b\,dy$ is a typical example.

Lemma 1.8. *Let f be a C^∞-function defined on a domain $U \subset \mathbb{R}^2$. Then $d(df) = 0$ holds.*

Proof. $d(df) = d(f_x\,dx + f_y\,dy) = (f_{yx} - f_{xy})\,dx \wedge dy = 0.$ □

Theorem 1.9 (Poincaré lemma)**.** *Let U be a simply connected domain, and α a differential 1-form defined on U. If $d\alpha = 0$, then there exists a C^∞ function f defined on U such that $df = \alpha$.*

The definition, fundamental properties of simple connectedness will be given in Section 3.

Exercises

1-1 Let $f(x,y) = e^{ax}\cos y$, where a is a constant. Find a function $g(x,y)$ satisfying

$$g_x = -f_y, \qquad g_y = f_x, \qquad g(0,0) = 0.$$

1-2 Let $U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\}$ and consider a 1-form

$$\alpha = a(x,y)\,dx + b(x,y)\,dy := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

on U. Take a point $\mathrm{P} = (r\cos\theta, r\sin\theta) \in U$ ($r > 1$, $0 < \theta < \pi$), and two curves

$$\begin{aligned} c_1(t) &:= \bigl(x_1(t), y_1(t)\bigr) = (\cos t, \sin t) && (0 \leqq t \leqq \theta),\\ c_2(s) &:= \bigl(x_2(s), y_2(s)\bigr) = (s\cos\theta, s\sin\theta) && (1 \leqq s \leqq r), \end{aligned}$$

whose union gives a curve joining $(1,0)$ and P. Compute the line integral

$$\begin{aligned} \int_{c_1\cup c_2} \alpha := &\int_0^\theta \left(a(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + b(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt \right)\\ &+ \int_1^r \left(a(x_2(s), y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds \right). \end{aligned}$$

2 Ordinary Differential Equations

The fundamental theorem for ordinary differential equations.

Consider a function

$$\boldsymbol{f}\colon I\times U \ni (t,\boldsymbol{x}) \longmapsto \boldsymbol{f}(t,\boldsymbol{x}) \in \mathbb{R}^m \tag{2.1}$$

of class C^1, where $I\subset\mathbb{R}$ is an interval and $U\subset\mathbb{R}^m$ is a domain in the Euclidean space $\mathbb{R}^m$. For any fixed $t_0\in I$ and $\boldsymbol{x}_0\in U$, the condition

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{f}\bigl(t,\boldsymbol{x}(t)\bigr), \qquad \boldsymbol{x}(t_0)=\boldsymbol{x}_0 \tag{2.2}$$

of an $\mathbb{R}^m$-valued function $t\mapsto \boldsymbol{x}(t)$ is called the *initial value problem of ordinary differential equation*, *ODE* for short, for unknown function $\boldsymbol{x}(t)$. For a subinterval J of I with $t_0\in I$, a function $\boldsymbol{x}\colon J\to U$ satisfying (2.2) is called a *solution* of the initial value problem.

Fact 2.1 (The existence theorem for ODE's)**.** *Let $\boldsymbol{f}\colon I\times U\to\mathbb{R}^m$ be a C^1-function as in* (2.1)*. Then, for any $\boldsymbol{x}_0\in U$ and $t_0\in I$, there exists a positive number ε and a C^1-function $\boldsymbol{x}\colon I\cap(t_0-\varepsilon,t_0+\varepsilon)\to U$ satisfying* (2.2)*.*

Take two solutions $\boldsymbol{x}_j\colon J_j\to U$ $(j=1,2)$ of (2.2) defined on subintervals $J_j\subset I$ containing t_0. Then the function $\boldsymbol{x}_2$ is said to be an *extension* of $\boldsymbol{x}_1$ if $J_1\subset J_2$ and $\boldsymbol{x}_2(t)=\boldsymbol{x}_1(t)$ for all $t\in J_1$. A solution $\boldsymbol{x}$ of (2.2) is said to be *maximal* if there are no non-trivial extension of it.

Fact 2.2 (The uniqueness for ODE's)**.** *The maximal solution of* (2.2) *is unique.*

Fact 2.3 (Smoothness of the solutions)**.** *If $\boldsymbol{f}\colon I\times U\to\mathbb{R}^m$ is of class C^r $(r=1,\dots,\infty)$, the solution of* (2.2) *is of class C^{r+1}. Here, $\infty+1=\infty$, as a convention.*

Let $V\subset\mathbb{R}^k$ be another domain of $\mathbb{R}^k$ and consider a C^∞-function

$$\boldsymbol{h}\colon I\times U\times V\ni(t,\boldsymbol{x};\boldsymbol{\alpha})\mapsto \boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha})\in\mathbb{R}^m. \tag{2.3}$$

For fixed $t_0\in I$, we denote by $\boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ the (unique, maximal) solution of (2.2) for $\boldsymbol{f}(t,\boldsymbol{x})=\boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha})$. Then

Fact 2.4. *The map $(t,\boldsymbol{x}_0;\boldsymbol{\alpha})\mapsto\boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ is of class C^∞.*

Example 2.5. (1) Let $m=1$, $I=\mathbb{R}$, $U=\mathbb{R}$ and $f(t,x)=\lambda x$, where λ is a constant. Then $x(t)=x_0\exp(\lambda t)$ defined on $\mathbb{R}$ is the maximal solution to

$$\frac{d}{dt}x(t)=f(t,x(t))=\lambda x(t), \qquad x(0)=x_0.$$

(2) Let $m=2$, $I=\mathbb{R}$, $U=\mathbb{R}^2$ and $\boldsymbol{f}(t;(x,y))=(y,-\omega^2x)$, where ω is a constant. Then

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}x_0\cos\omega t+\frac{y_0}{\omega}\sin\omega t\\ -x_0\omega\sin\omega t+y_0\cos\omega t\end{pmatrix}$$

is the unique solution of

$$\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}y(t)\\ -\omega^2x(t)\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix}=\begin{pmatrix}x_0\\ y_0\end{pmatrix},$$

defined on $\mathbb{R}$. This equation can be considered as a single equation

$$\frac{d^2}{dt^2}x(t)=-\omega^2x(t), \quad x(0)=x_0, \quad \frac{dx}{dt}(0)=y_0$$

of order 2.

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(3) Let $m = 1$, $I = \mathbb{R}$, $U = \mathbb{R}$ and $f(t,x) = t(1+x^2)$. Then $x(t) = \tan\frac{t^2}{1}$ defined on $(-\sqrt{\pi}, \sqrt{\pi})$ is the unique maximal solution of the initial value problem

$$\frac{dx}{dt} = t(1+x^2), \qquad x(0) = 0.$$

Linear Ordinary Differential Equations.

The ordinary differential equation (2.2) is said to be *linear* if the function (2.1) is a linear function in $\boldsymbol{x}$, that is, a linear differential equation is in a form

$$\frac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

where $A(t)$ and $\boldsymbol{b}(t)$ are $m \times m$-matrix-valued and $\mathbb{R}^m$-valued functions in t, respectively.

For the sake of later use, we consider, in this lecture, the special form of linear differential equation for matrix-valued unknown functions as follows: Let $\mathrm{M}_n(\mathbb{R})$ be the set of $n \times n$-matrices with real components, and take functions

$$\varOmega\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}), \quad \text{and} \qquad B\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}),$$

where $I \subset \mathbb{R}$ is an interval. Identifying $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, we assume $\varOmega$ and B are continuous functions (with respect to the topology of $\mathbb{R}^{n^2} = \mathrm{M}_n(\mathbb{R})$). Then we can consider the linear ordinary differential equation for matrix-valued unknown $X(t)$ as

$$\frac{dX(t)}{dt} = X(t)\varOmega(t) + B(t), \qquad X(t_0) = X_0, \tag{2.4}$$

where X_0 is given constant matrix.

Then, the fundamental theorem of *linear* ordinary equation states that *the maximal solution of* (2.4) *is defined on whole* I. To prove this, we prepare some materials related to matrix-valued functions.

Preliminaries: Matrix Norms.

Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n \times n$-matrices with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$|X|_{\mathrm{E}} = \sqrt{\operatorname{tr}(X^T X)} = \sqrt{\sum_{i,j=1}^{n} x_{ij}^2} \tag{2.5}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$|X|_{\mathrm{M}} := \sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\, \boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\right\}, \tag{2.6}$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$.

Lemma 2.6. (1) *The map* $X \mapsto |X|_{\mathrm{M}}$ *is a norm of* $\mathrm{M}_n(\mathbb{R})$.

(2) *For* X, $Y \in \mathrm{M}_n(\mathbb{R})$, *it holds that* $|XY|_{\mathrm{M}} \leqq |X|_{\mathrm{M}}\,|Y|_{\mathrm{M}}$.

(3) *Let* $\lambda = \lambda(X)$ *be the maximum eigenvalue of semi-positive definite symmetric matrix* $X^T X$. *Then* $|X|_{\mathrm{M}} = \sqrt{\lambda}$ *holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}} \leqq |X|_{\mathrm{M}} \leqq |X|_{\mathrm{E}}$.

(5) *The map* $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ *is continuous with respect to the Euclidean norm.*

Proof. Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}} = \sup\{|X\boldsymbol{v}|\,;\,\boldsymbol{v} \in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Since $S^{n-1} \ni \boldsymbol{x} \mapsto |A\boldsymbol{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, it takes the maximum. Thus, the right-hand side of (2.6) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm[4].

Since $A := X^T X$ is positive semi-definite, its eigenvalues λ_j $(j = 1,\dots,n)$ are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j = \lambda_j \boldsymbol{a}_j$ $(j = 1,\dots,n)$. Let λ be the maximum eigenvalue of A, and write $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \cdots + v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v}, X\boldsymbol{v}\rangle = \lambda_1 v_1^2 + \cdots + \lambda_n v_n^2 \leqq \lambda\,\langle \boldsymbol{v}, \boldsymbol{v}\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the λ-eigenvector, proving (3). Noticing that the norm (2.5) is invariant under conjugations $X \mapsto P^T X P$ $(P \in \mathrm{O}(n))$, we obtain $|X|_{\mathrm{E}} = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$ by diagonalizing $X^T X$ by an orthogonal matrix P. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology as $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). $\square$

Preliminaries: Matrix-valued Functions.

Lemma 2.7. *Let* X *and* Y *be* C^∞*-maps defined on a domain* $U \subset \mathbb{R}^m$ *into* $\mathrm{M}_n(\mathbb{R})$*. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY) = \dfrac{\partial X}{\partial u_j}Y + X\dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X = \operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1} = -X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

where $\widetilde{X}$ *is the cofactor matrix of* X*, and we assume in* (3) *that* X *is a regular matrix.*

Proof. The formula (1) holds because the definition of matrix multiplication and the Leibnitz rule, Denoting $' = \partial/\partial u_j$,

$$O = (\mathrm{id})' = (X^{-1}X)' = (X^{-1})X' + (X^{-1})'X$$

implies (3), where id is the identity matrix.

Decompose the matrix X into column vectors as $X = (\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)$. Since the determinant is multi-linear form for n-tuple of column vectors, it holds that

$$(\det X)' = \det(\boldsymbol{x}_1', \boldsymbol{x}_2, \dots, \boldsymbol{x}_n) + \det(\boldsymbol{x}_1, \boldsymbol{x}_2', \dots, \boldsymbol{x}_n) + \cdots + \det(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n').$$

Then by cofactor expansion of the right-hand side, we obtain (2). $\square$

Proposition 2.8. *Assume two* C^∞ *matrix-valued functions* $X(t)$ *and* $\varOmega(t)$ *satisfy*

$$\frac{dX(t)}{dt} = X(t)\varOmega(t), \qquad X(t_0) = X_0. \tag{2.7}$$

Then

$$\det X(t) = (\det X_0)\exp\int_{t_0}^{t}\operatorname{tr}\varOmega(\tau)\,d\tau \tag{2.8}$$

holds. In particular, if $X_0 \in \mathrm{GL}(n,\mathbb{R})$*,*[5] *then* $X(t) \in \mathrm{GL}(n,\mathbb{R})$ *for all* t*.*

[4] $|X|_{\mathrm{M}} > 0$ whenever $X \neq O$, $|\alpha X|_{\mathrm{M}} = |\alpha|\,|X|_{\mathrm{M}}$, and the triangle inequality $|X+Y|_{\mathrm{M}} \leqq |X|_{\mathrm{M}} + |Y|_{\mathrm{M}}$.

[5] $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,\det A \neq 0\}$: the general linear group.

Proof. By (2) of Lemma 2.7, we have

$$\begin{aligned}\frac{d}{dt}\det X(t) &= \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)\\ &= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).\end{aligned}$$

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (2.8). □

Corollary 2.9. *If $\Omega(t)$ in (2.7) satisfies $\operatorname{tr}\Omega(t) = 0$, then $\det X(t)$ is constant. In particular, if $X_0 \in \mathrm{SL}(n,\mathbb{R})$, X is a function valued in $\mathrm{SL}(n,\mathbb{R})$* [6].

Proposition 2.10. *Assume $\Omega(t)$ in (2.7) is skew-symmetric for all t, that is, $\Omega^T + \Omega$ is identically O. If $X_0 \in \mathrm{O}(n)$ (resp. $X_0 \in \mathrm{SO}(n)$)*[7]*, then $X(t) \in \mathrm{O}(n)$ (resp. $X(t) \in \mathrm{SO}(n)$) for all t.*

Proof. By (1) in Lemma 2.7,

$$\begin{aligned}\frac{d}{dt}(XX^T) &= \frac{dX}{dt}X^T + X\left(\frac{dX}{dt}\right)^T\\ &= X\Omega X^T + X\Omega^T X^T = X(\Omega + \Omega^T)X^T = O.\end{aligned}$$

Hence XX^T is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)X(t)^T = X(t_0)X(t_0)^T = X_0X_0^T = \mathrm{id}\,.$$

If $X_0 \in \mathrm{O}(n)$, this proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. □

Preliminaries: Norms of Matrix-Valued functions.

Let $I = [a,b]$ be a closed interval, and denote by $C^0(I, \mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X\colon I \to \mathrm{M}_n(\mathbb{R})$. For any positive number k, we define

$$||X||_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\, t \in I\right\} \tag{2.9}$$

for $X \in C^0(I, \mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $||\cdot||_{I,0}$ is the *uniform norm* for continuous functions, which is complete. Similarly, one can prove the following in the same way:

Lemma 2.11. *The norm $||\cdot||_{I,k}$ on $C^0(I, \mathrm{M}_n(\mathbb{R}))$ is complete.*

Linear Ordinary Differential Equations.

We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 2.12. *Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,. \tag{2.10}$$

[6] $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, \det A = 1\}$; the special lienar group.

[7] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, A^TA = AA^T = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\, \det A = 1\}$: the special orthogonal group.

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (2.10). Then

$$Y(t)-X(t)=\int_{t_0}^{t}\bigl(Y'(\tau)-X'(\tau)\bigr)\,d\tau=\int_{t_0}^{t}\bigl(Y(\tau)-X(\tau)\bigr)\Omega(\tau)\,d\tau \qquad \left({}'=\frac{d}{dt}\right)$$

holds. Take an arbitrary closed interval $J\subset I$. Then for an arbitrary $t\in J$,

$$\begin{aligned}|Y(t)-X(t)|_{\mathrm{M}}&\leqq\left|\int_{t_0}^{t}\left|\bigl(Y(\tau)-X(\tau)\bigr)\Omega(\tau)\right|_{\mathrm{M}}d\tau\right|\leqq\left|\int_{t_0}^{t}|Y(\tau)-X(\tau)|_{\mathrm{M}}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right|\\&=\left|\int_{t_0}^{t}e^{-k\tau}\,|Y(\tau)-X(\tau)|_{\mathrm{M}}\,e^{k\tau}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right|\leqq||Y-X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\left|\int_{t_0}^{t}e^{k\tau}\,d\tau\right|\\&=||Y-X||_{J,k}\frac{\sup_J|\Omega|_{\mathrm{M}}}{|k|}e^{kt}\left|1-e^{-k(t-t_0)}\right|\leqq||Y-X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\frac{e^{kt}}{|k|}\end{aligned}$$

holds, and hence

$$e^{-kt}|Y(t)-X(t)|_{\mathrm{M}}\leqq\frac{\sup_J|\Omega|_{\mathrm{M}}}{|k|}||Y-X||_{J,k}.$$

Thus, for an appropriate choice of $k\in\mathbb{R}$, it holds that

$$||Y-X||_{J,k}\leqq\frac{1}{2}||Y-X||_{J,k},$$

that is, $||Y-X||_{J,k}=0$, proving $Y(t)=X(t)$ for $t\in J$. Since J is arbitrary, $Y=X$ holds on I.
Existence: Take $a>0$such that $J:=[t_0,a]\subset I$, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t)=\mathrm{id}$ and

$$X_{j+1}(t)=\mathrm{id}+\int_{t_0}^{t}X_j(\tau)\Omega(\tau)\,d\tau \quad (j=0,1,2,\dots). \tag{2.11}$$

Then

$$\begin{aligned}|X_{j+1}(t)-X_j(t)|_{\mathrm{M}}&\leqq\int_{t_0}^{t}|X_j(\tau)-X_{j-1}(\tau)|_{\mathrm{M}}|\Omega(\tau)|_{\mathrm{M}}\,d\tau\\&\leqq\frac{e^{k(t-t_0)}}{|k|}\sup_J|\Omega|_{\mathrm{M}}||X_j-X_{j-1}||_{J,k},\end{aligned}$$

and hence $||X_{j+1}-X_j||_{J,k}\leqq\frac{1}{2}||X_j-X_{j-1}||_{J,k}$, for an appropriate choice of $k\in\mathbb{R}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $||\cdot||_{J,k}$. Thus, by completeness (Lemma 2.11), it converges to some $X\in C^0(J,\mathrm{M}_n(\mathbb{R}))$. By (2.11), the limit X satisfies

$$X(t_0)=\mathrm{id},\qquad X(t)=\mathrm{id}+\int_{t_0}^{t}X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t)=X(t)\Omega(t)$ $({}'=d/dt)$. By the same argument for $a<t_0$ with $J=[a,t_0]$, existence of the solution on I is proven.

Finally, we shall prove that X is of class C^∞. Since $X'(t)=X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1, and then X is of class C^2. Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r. □

Corollary 2.13. *Let $\Omega(t)$ be a matrix-valued C^∞-function defined on an interval I. Then for each $t_0\in I$ and $X_0\in\mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that*

$$\frac{dX(t)}{dt}=X(t)\Omega(t),\quad X(t_0)=X_0\quad \bigl(X(t):=X_{t_0,X_0}(t)\bigr) \tag{2.12}$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Proof. We rewrite $X(t)$ in Proposition 2.12 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{2.13}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 2.8,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\frac{dW}{dt} = \frac{dX}{dt}Y^{-1} - XY^{-1}\frac{dY}{dt}Y^{-1} = X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O,$$

that is, W is constant, and hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

So the uniqueness is obtained. The final part is obvious by the expression (2.13). □

Proposition 2.14. *Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{2.14}$$

Proof. Rewrite X in Proposition 2.12 as $Y := X_{t_0,\mathrm{id}}$. Then

$$X(t) = \left(X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{2.15}$$

satisfies (2.14). Conversely, if X satisfies (2.14), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (2.15). □

Theorem 2.15. *Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{2.16}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

Proof. Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (2.16) is equivalent to

$$\frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0, \tag{2.17}$$

where $\tilde{\boldsymbol{\alpha}} := (t_0, \alpha_1, \dots, \alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{0,X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (2.17) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 2.14. So it is sufficient to show differentiability with respect to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ the unique solution of

$$\frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O. \tag{2.18}$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$. In particular, by the proof of Proposition 2.14, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right) Y^{-1}(\tau)d\tau\right) Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a C^∞-function in $(t,\tilde{\boldsymbol{\alpha}})$. □

An Application: Fundamental Theorem for Space Curves.

A C^∞-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \subset \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot\gamma \neq \mathbf{0}$ holds on I. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde\gamma(s) := \gamma(t(s))$ satisfies $|\tilde\gamma'(s)| = 1$. Such a parameter s is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(s)\times\boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e},\boldsymbol{n},\boldsymbol{b}\}$ of $\mathbb{R}^3$ for each s. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s),\boldsymbol{n}(s),\boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{2.19}$$

in s, which is called the *Frenet frame* associated to the curve γ. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s)\right\rangle,$$

which are called the *curvature* and *torsion*, respectively, of γ. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{2.20}$$

Proposition 2.16. *The curvature and the torsion are invariant under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ of $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.*

Proof. Let κ, τ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of γ_1, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (2.20), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 and γ_2 have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (2.20). Let $\mathcal{F}$ be the unique solution of (2.20) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 2.13, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ $(j = 1, 2)$. In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ $(A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3))$. Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

Theorem 2.17 (The fundamental theorem for space curves)**.**
Let $\kappa(s)$ and $\tau(s)$ be C^∞-functions defined on an interval I satisfying $\kappa(s) > 0$ on I. Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively. Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ $(A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3)$ of $\mathbb{R}^3$.

Proof. We have already shown the uniqueness in Proposition 2.16. We shall prove the existence: Let $\Omega(s)$ be as in (2.20), and $\mathcal{F}(s)$ the solution of (2.20) with $\mathcal{F}(s_0) = \mathrm{id}$. Since Ω is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 2.10. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the Frenet frame of γ, and κ, and τ are the curvature and torsion of γ, respectively. □

Exercises

2-1 Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1 - x), \qquad x(0) = a,$$

where a is a real number.

2-2 Let $x = x(t)$ be the maximal solution of an initial value problem of differential equation

$$\frac{d^2x}{dt^2} = -\sin x, \qquad x(0) = 0, \quad \frac{dx}{dt}(0) = 2.$$

- Show that $\dfrac{dx}{dt} = 2\cos\dfrac{x}{2}$.
- Verify that x is defined on $\mathbb{R}$, and compute $\lim_{t\to\pm\infty} x(t)$.

2-3 Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature κ and torsion τ satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1 + s^2)}.$$

Bibliography

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Glossary

1-form 微分 1-形式, 3

arc-length parameter 弧長径数, 11

column vector 列ベクトル, 1, 7
commutativity 可換性, 2
curvature 曲率, 11

determinant 行列式, 1
differential 2-form 微分 2-形式, 3
differential form 微分形式, 3
differential one form 微分 1-形式, 3
differential 微分, 3
distance 距離, 1
domain 領域, 1

eigenvalue 固有値, 7
Euclidean space ユークリッド空間, 1
exterior differential 外微分, 3

Frenet frame フルネ枠, 11

general linear group ($\mathrm{GL}(n, \mathbb{R})$) 一般線形群, 7

identity matrix 単位行列, 1
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isometry 等長写像，等長変換, 1

latitude 緯度, 2
linear function 1 次関数, 6
linear ordinary differential equation 線形常微分方程式, 6
longitude 経度, 2

map 写像, 1
matrix 行列, 1
mean value theorem 平均値の定理, 2
Mercator's world map メルカトルの世界地図, 2

norm ノルム, 1

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orientation preserving 向きを保つ, 1
orthogonal group ($\mathrm{O}(n)$) 直交群, 8
orthogonal matrix 直交行列, 1

parametrization パラメータ表示, 2
partial derivative 偏微分，偏導関数, 2
Poincaré lemma ポアンカレの補題, 4

regular curve 正則曲線, 11
regular matrix 正則行列, 7
row vector 行ベクトル, 1

simply connected 単連結, 4
skew-symmeetric matrix 交代行列，歪対称行列, 8
solution 解, 5
space curve 空間曲線, 11
special linear group ($\mathrm{SL}(n, \mathbb{R})$) 特殊線形群, 8
special orthogonal group ($\mathrm{SO}(n)$) 特殊直交群, 8
sphere 球面, 2
surface 曲面, 1

torsion 捩率, 11
total differential 全微分, 3
transposition 転置, 1

unknown function 未知関数, 5