Advanced Topics in Geometry B1 (MTH.B406)

Asymptotic Chebyshev nets

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Exercise 2-1

- $K=-1 \Leftrightarrow$ partial diff. eq. (difficult)
- To find simple examples, killing 1-parameter is useful.

Problem

Let $\gamma(t)=\bigl(x(t),z(t)\bigr)$ *(*$\gamma\in I$*) be a parametrized curve on the* xz*-plane satisfying*

(trajectory of isometries (in $\mathbb{R}^3$))

$$\bigl(x'(t)\bigr)^2+\bigl(z'(t)\bigr)^2=1 \qquad (t\in I), \quad x>0 \qquad (*)$$

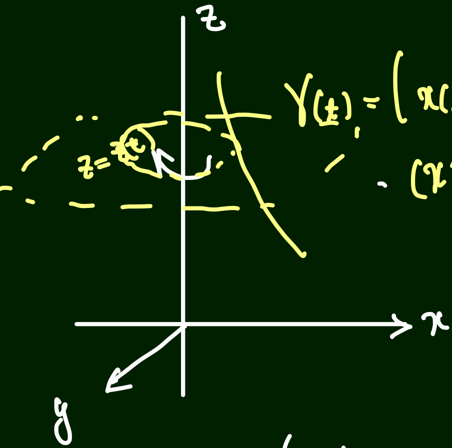
where $I\subset\mathbb{R}$ *is an interval. Consider a surface*

$$p_\gamma(s,t):=\bigl(x(t)\underline{\cos s},\,x(t)\underline{\sin s},\,z(t)\bigr),$$

$$\begin{pmatrix}\cos s & -\sin s & 0\\ \sin s & \cos s & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ 0\\ z\end{pmatrix}$$

which is a <u>*surface of revolution*</u> *of* <u>*profile curve*</u> γ*.*

1. *Show that* p_γ *is pseudospherical if and only if* $\underline{x''=x}$ *holds.*
2. *Can one choose* $I=\mathbb{R}$*?*



$\gamma(t) = \big(x(t), z(t) \big)$, "general"

$(x')^2 + (y')^2 = 1$ (unit speed)

t: arclength parameter

<u>Fact</u> $\forall$ regular curves can be reparametrized by the arclength param.

$$P = P_f = \big(x(t)\cos s,\ x(t)\sin s,\ z(t) \big)$$

the surface of revolution with profile curve γ.

$\underline{K=-1} \iff x'' = x$

$e_1 = (\cos S, \sin S, 0)$

$e_2 = (-\sin S, \cos S, 0)$

$e_3 = (0, 0, 1)$

$\because$ $p = x(t)\, e_1 + z(t)\, e_3$

- $p_s = x(t)\, e_2$
- $p_t = x' e_1 + z' e_3$
- $\nu = -z' e_1 + x' e_3$, $(|\nu|^2 = (z')^2 + (x')^2 = 1)$

$\nu_s = -z' e_2$

$\nu_t = -z'' e_1 + x'' e_3$

$$\Rightarrow \hat{I} = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{II} = \begin{pmatrix} -p_s \cdot \nu_s & -p_s \cdot \nu_t \\ -p_t \cdot \nu_s & -p_t \cdot \nu_t \end{pmatrix}$$

$$= \begin{pmatrix} -xz' & 0 \\ 0 & -x'z'' + x''z' \end{pmatrix}$$

$$K = \frac{\det \hat{\mathrm{II}}}{\det \hat{\mathrm{I}}} = \frac{\cancel{x} z'(x'z'' - x''z')}{x^{\cancel{2}}}$$

$$(x')^2 + (z')^2 = 1$$

$$2x'x'' + 2z'z'' = 0$$

$$= \frac{x'z'z'' - x''z'^2}{x}$$

$$= \frac{x'(-x'x'') - x''z'^2}{x}$$

$$= \frac{x''(-x'^2 - z'^2)}{x} = \frac{-x''}{x}$$

$$K = -1$$

$$\Leftrightarrow x'' = x$$

Exercise 2-1

(e^t, e^{-t})

$$x'' = x \Rightarrow x = A\cosh t + B\sinh t:$$

composition formula of hyperbolic fct's

- $\underline{A^2 - B^2 > 0}$, $x = \pm\sqrt{A^2 - B^2}\cosh(t + \alpha)$,
- $\underline{A^2 - B^2 < 0}$, $x = \pm\sqrt{B^2 - A^2}\sinh(t + \alpha)$,
- $\underline{A^2 = B^2}$, $x = \pm A e^{\pm t} = \pm e^{\pm t + \alpha}$

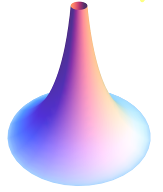
parameter change $(\pm t + \alpha \to t)$

$$x = \begin{cases} a\cosh t \\ a\sinh t \\ \underline{a e^{-t}} \end{cases}$$

Exercise 2-1

Beltrami's pseudosphere.

$x = e^{-\cancel{s}\,t}$



$$\underline{x'^2 + z'^2} = 1 \qquad \boxed{(t>0)}$$

$$z'^2 = 1 - e^{-2t}$$

$$z' = \cancel{\pm}\,+\sqrt{1-e^{-2t}}$$

$(z \mapsto -z)$

$$z = \int_0^t \sqrt{1-e^{-2u}}\,du$$

(up to additive const)

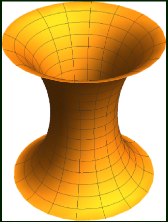
vertical translation

$$\underline{z = -\sqrt{1-e^{-2t}} + \cosh^{-1} e^{t}}$$

elementary fct.

Exercise 2-1

$x = A\cosh t$



$$z' = \sqrt{1 - A^2 \sinh^2 t}$$

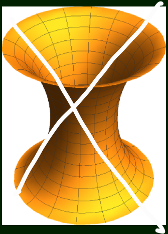
$$\sinh^2 \underline{\underline{t}} < \frac{1}{A^2}$$

$$z = \int \cdots$$

not an elementary fct.
(elliptic integrals)

Exercise 2-1

$x = A \sinh \cancel{s}\ t$



- $z' = \sqrt{1 - A^2 \cosh^2 t}$

$\cosh t \geq 1$

- $0 < A < 1$

Exercise 2-1

2. Can one choose $I = \mathbb{R}$? No.

$$\begin{cases} x' \longrightarrow \infty \\ (x')^2 + (z')^2 = 1 \end{cases}$$

Resulting surfaces have singularities.

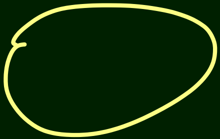
$\exists$ example w/o singularities?

Ans No (Hilbert, 1900's) Lecture 5.

Exercise 2-1

Q: Does the fact that I can only be defined on a finite interval mean that the looped $\gamma(t)$ cannot be? I might be wrong.

$\gamma(t)$



taking the universal cover of the loop

$\rightarrow$ defined on $\mathbb{R}$.

Exercise 2-2

Problem

Let a and b be real numbers with $a \neq 0$. Compute the Gaussian curvature of the surface

pseudosphere

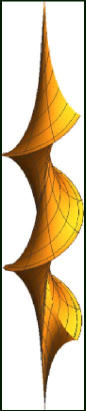
$$p(u, v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0, 0, u).$$

rotation parameter

helical surface

$$\begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

Exercise 2-2



Dini's pseudosphere

$$ds^2 = (a^2 \operatorname{sech}^2 v + b^2)\,du^2 + 2ab \operatorname{th}^2 v \, du\, dv + a^2 \operatorname{th}^2 v \, dv^2$$

$$\mathrm{II} = -\operatorname{th} v \operatorname{sech} v \times (a^2 du^2 - 2ab\, du\, dv + b^2 dv^2)$$

$$K = \frac{-1}{a^2 + b^2}$$