· (General construction of pseudospherical surfaces.)

Advanced Topics in Geometry B1 (MTH.B406)

Asymptotic Chebyshev nets

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· Choose parameters of surfaces.

Today's Goal

$\det \hat{I} = 1 - \cos^2\theta = \sin^2\theta > 0$

$(\theta \not\equiv 0 \mod \pi)$

Theorem (Theorem 3.9)

For a each point P of a surface of constant negative Gaussian curvature $-k^2$, there exists a neighborhood U of P and coordinate system (ξ, η) such that the first and second fundamental forms are in the form

- $ds^2 = \overset{1}{d\xi^2} + 2\cos\theta\, d\xi\, d\eta + \overset{1}{d\eta^2}, \qquad II = 2k\sin\theta\, d\xi\, d\eta,$

where θ is a smooth function in (ξ, η) with $\boxed{0 < \theta(\xi, \eta) < \pi.}$

The asymptotic Chebyshev net.

anti-diagonal

$$\hat{I} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} \qquad \hat{II} = \begin{pmatrix} 0 & k\sin\theta \\ k\sin\theta & 0 \end{pmatrix}$$

$\det \hat{II} = -k^2\sin^2\theta < 0$

Asymptotic Coordinate system

$p(u,v)$: a regular parametrization of a surface in $\mathbb{R}^3$:

- (u,v) is an asymptotic coordinate system $\Leftrightarrow II = 2M\,du\,dv$

Proposition (Asymptotic Coordinate system; Prop. 3.8)

Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface in $\mathbb{R}^3$ whose Gaussian curvature is negative on U. Then for each $\mathrm{P} \in U$, there exists an asymptotic coordinate system on a neighborhood of P.

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

$K<0$

$\Rightarrow \underbrace{LN - M^2}_{-k^2} < 0$

We may assume (without loss of generality) that $L \neq 0$ at $\underline{\underline{P}}$.

☺ · $L=0$ & $N \neq 0 \Rightarrow (u,v) \mapsto (v,u) \Rightarrow L \neq 0$

· $L=N=0$; $M \neq 0$

$u = s+t$

$v = s-t$

$$\mathrm{II} = 2M(ds^2 - dt^2) \text{ at } P$$

$$\mathrm{II} = L\left(du^2 + \frac{2M}{L}du\,dv + \frac{M^2}{L^2}dv^2 - \left(\frac{M^2}{L^2} - \frac{N}{L}\right)dv^2\right)$$

$$= L\left(\left(du + \frac{M}{L}dv\right)^2 - \frac{M^2-LN}{L^2}dv^2\right)$$

$$= L\left(\left(du + \frac{M}{L}dv\right)^2 - \left(\frac{k}{L}\right)^2 dv^2\right)$$

$$= \mathrm{II} = L\left(\overbrace{(\alpha\,du + \beta\,dv)}^{d\xi}\,\overbrace{(\gamma\,du + \delta\,dv)}^{d\eta} \right)$$

$$d\xi = \varphi\,\underline{(\alpha\,du + \beta\,dv)}$$

not closed in general

$$d\eta = \psi\,(\gamma\,du + \delta\,dv)$$

integrating factors

Integrating factor

Lemma (A special case of Caratheodory's)

Let $\omega = \alpha\, du + \beta\, dv$ be a 1-form defined on a domain U of the uv-plane $\mathbb{R}^2$, where $\overset{\alpha}{\cancel{\mathrm{R}}}$ and $\overset{\beta}{\cancel{\mathrm{Q}}}$ are functions in (u, v). Assume $(\alpha, \beta) \neq (0, 0)$ at $\mathrm{P} \in U$. Then there exists a neighborhood $V \subset U$ of P and functions φ and ξ on V such that

$$\varphi\omega = d\xi, \qquad \varphi(\mathrm{Q}) \neq 0 \quad \text{for} \quad \mathrm{Q} \in V.$$

" $\exists \varphi$ s.t. $\varphi\omega = d\xi \Leftrightarrow \omega \wedge d\omega = 0$ "

Proof of existence of Asymp. Chebyshev net

- $ds^2 = \overset{1}{E}\, du^2 + 2F\, du\, dv + \overset{1}{G}\, dv^2$

  $\mathrm{II} = 2M\, du\, dv$

$\Rightarrow$ Apply Problem 5-1 in 1Q

$$E_v = G_u = 0 \quad \leftarrow \text{Codazzi eq}$$

$$E = (e(u))^2 \quad G = (g(v))^2$$

$$E\, du^2 = (e\, du)^2 = dx^2 \qquad G\, dv^2 = dy^2$$

$$x = \int e\, du \qquad y = \int g\, dv$$

$\underline{\theta = \theta(\xi, \eta)}$

$$p(u,v) = (\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v)$$
$$ds^2 = \operatorname{sech}^2 v\, du^2 + \tanh^2 v\, dv^2,$$
$$II = -\operatorname{sech} v \tanh v (du^2 - dv^2).$$

$= -\operatorname{sech} v \tanh v\, (du - dv)(du + dv)$

$d\xi = du - dv \qquad \xi = u - v$

$d\eta = du + dv \qquad \eta = u + v$

$u = \frac{1}{2}(\xi + \eta)$

$v = \frac{1}{2}(-\xi + \eta)$

Exercise 3-1

Problem

Let a and b be real numbers with $a \neq 0$ and

$$p(u,v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0,0,u).$$

Find a coordinate change $(u,v) \mapsto (\xi,\eta)$ to an asymptotic Chebyshev net for p, and give an explicit expression of θ as a function in (ξ,η).

Exercise 3-2 · (uniqueness)

Let (ξ, η) be an asymptotic Chebyshev net on a surface. Assume another parameter (x, y) is also an asymptotic Chebyshev net. Prove that (x, y) satisfies

$$(x, y) = (\pm\xi + x_0, \pm\eta + y_0) \quad \text{or} \quad (x, y) = (\pm\eta + x_0, \pm\xi + y_0)$$

where x_0 and y_0 are constants.