Advanced Topics in Geometry B1 (MTH.B406)

Asymptotic Chebyshev nets

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Today's Goal

Theorem (Theorem 3.9)

For a each point P of a surface of constant negative Gaussian curvature $-k^2$, there exists a neighborhood U of P and coordinate system (ξ, η) such that the first and second fundamental forms are in the form

$$ds^2 = d\xi^2 + 2\cos\theta\, d\xi\, d\eta + d\eta^2, \qquad II = 2k\sin\theta\, d\xi\, d\eta,$$

where θ is a smooth function in (ξ, η) with $0 < \theta(\xi, \eta) < \pi$.

The asymptotic Chebyshev net.

Asymptotic Coordinate system

$p(u,v)$: a regular parametrization of a surface in $\mathbb{R}^3$:

- (u,v) is an asymptotic coordinate system $\Leftrightarrow II = 2M\,du\,dv$

Proposition (Asymptotic Coordinate system; Prop. 3.8)

Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface in $\mathbb{R}^3$ whose Gaussian curvature is negative on U. Then for each $\mathrm{P} \in U$, there exists an asymptotic coordinate system on a neighborhood of P.

Integrating factor

Lemma (A special case of Caratheodory's)

Let $\omega = \alpha\, du + \beta\, dv$ *be a* 1*-form defined on a domain* U *of the* uv*-plane* $\mathbb{R}^2$*, where* P *and* Q *are functions in* (u, v)*. Assume* $(\alpha, \beta) \neq (0, 0)$ *at* $\mathrm{P} \in U$*. Then there exists a neighborhood* $V \subset U$ *of* P *and functions* φ *and* ξ *on* V *such that*

$$\varphi\omega = d\xi, \qquad \varphi(\mathrm{Q}) \neq 0 \quad \textit{for} \quad \mathrm{Q} \in V.$$

Example

$$p(u,v) = (\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v)$$
$$ds^2 = \operatorname{sech}^2 v\, du^2 + \tanh^2 v\, dv^2,$$
$$II = -\operatorname{sech} v \tanh v (du^2 - dv^2).$$

Exercise 3-1

Problem

Let a and b be real numbers with $a \neq 0$ and

$$p(u,v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0,0,u).$$

Find a coordinate change $(u,v) \mapsto (\xi,\eta)$ to an asymptotic Chebyshev net for p, and give an explicit expression of θ as a function in (ξ,η).

Exercise 3-2

Let (ξ, η) be an asymptotic Chebyshev net on a surface. Assume another parameter (x, y) is also an asymptotic Chebyshev net. Prove that (x, y) satisfies

$$(x, y) = (\pm\xi + x_0, \pm\eta + y_0) \quad \text{or} \quad (x, y) = (\pm\eta + x_0, \pm\xi + y_0)$$

where x_0 and y_0 are constants.