Advanced Topics in Geometry B1 (MTH.B406)

A construction of pseudospherical surfaces

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Exercise 3-1

Problem

Let a and b be real numbers with $a \neq 0$ and

Ex 2-2

Dini's p.s.

$$p(u,v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0,0,u).$$

Find a coordinate change $(u,v) \mapsto (\xi,\eta)$ to an asymptotic Chebyshev net for p, and give an explicit expression of θ as a function in (ξ,η).

$$K = -\frac{1}{a^2+b^2} < 0$$

Exercise 3-1

$1-\tanh^2 v$

$\mathrm{sech}^2 v + \tanh^2 v = 1$

$$\rho := \sqrt{a^2+b^2} > 0$$

$$\begin{aligned} ds^2 &= (a^2\,\mathrm{sech}^2\, v + b^2)\, du^2 + 2ab\tanh^2 v\, du\, dv + a^2\tanh^2 v\, dv^2 \\ &= \rho^2\, du^2 - \tanh^2 v(a^2\, du^2 - 2ab\, du\, dv - a^2\, dv^2) \\ II &= -\rho^{-1}\,\mathrm{sech}\, v\tanh v(a^2\, du^2 - 2ab\, du\, dv - a^2\, dv^2) \end{aligned}$$

Exercise 3-1

$\rho^2 = a^2 + b^2$

$+ b^2 dv^2 - b^2 dv^2$

- $$\begin{aligned} a^2\,du^2 - 2ab\,du\,dv - a^2\,dv^2 &= (a\,du - b\,dv)^2 - (\rho\,dv)^2 \\ &= \underbrace{(a\,du - (b+\rho)\,dv)}_{d\xi}\underbrace{(a\,du - (b-\rho)\,dv)}_{d\eta} \\ &= d\xi\,d\eta \end{aligned}$$

where

$$\xi = au - (b+\rho)v, \quad \eta = au - (b-\rho)v.$$

$$
\begin{aligned}
ds^2 &= \rho^2\, du^2 - \cancel{a} \tanh^2 v \, d\xi\, d\eta \\
II &= -\rho^{-1} \operatorname{sech} v \tanh v \, d\xi\, d\eta \qquad \xi
\end{aligned}
$$

$$
\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a & -(b+\rho) \\ a & -(b-\rho) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},
$$

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2a\rho} \begin{pmatrix} -(b-\rho) & (b+\rho) \\ -a & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.
$$

$$
ds^2 = \left(\underbrace{\frac{b-\rho}{2a} d\xi}_{dx}\right)^2 + \left(\frac{1}{2} - \tanh^2 v\right) d\xi\, d\eta + \left(\underbrace{\frac{b+\rho}{2a} d\eta}_{dy}\right)^2
$$

$$
= dx^2 + 2\cos\theta\, dx\, dy + dy^2
$$

Exercise 3-1

$\cos\theta = 2\tanh^2 v - 1$

$\sin\theta = 2\,\mathrm{sech}\,v\tanh v$

$$x = \frac{b-\rho}{2a}\xi, \qquad y = \frac{b+\rho}{2a}\eta \qquad \Rightarrow$$

$$\begin{aligned} ds^2 &= dx^2 + 2(2\tanh^2 v - 1)\,dx\,dy + dy^2 \\ &= dx^2 + 2\cos\theta\,dx\,dy + dy^2 \\ II &= \overset{2}{\cancel{4}}\rho^{-1}\,2\,\operatorname{sech} v\tanh v\,dx\,dy \\ &= 2\rho^{-1}\sin\theta\,dx\,dy \end{aligned}$$

Set

$$\cos\frac{\theta}{2} = \tanh v, \qquad \sin\frac{\theta}{2} = \operatorname{sech} v,$$

Exercise 3-1

$$\cos\frac{\theta}{2} = \tanh v, \quad \sin\frac{\theta}{2} = \operatorname{sech} v, \qquad v = \frac{1}{2\rho}\left(\frac{2a}{\rho - b}x + \frac{2a}{\rho + b}y\right)$$

$$= \frac{e^{v} - e^{-v}}{e^{v} + e^{-v}} = \frac{1 - e^{-2v}}{1 + e^{-2v}}$$

$$= \frac{2}{e^{v} + e^{-v}} = \frac{2e^{-v}}{1 + e^{-2v}}$$

$$e^{-v} = \tan\frac{\theta}{4}$$

$$\theta = 4\tan^{-1} e^{-v} = 4\tan^{-1}\left(\frac{-1}{\rho}\left(\mu x + \frac{1}{\mu}y\right)\right)$$

$$\theta = 4\tan^{-1}\exp\left\{\frac{-1}{\rho}\left(\mu x + \frac{1}{\mu}y\right)\right\}$$

if $\rho = 1$: $\theta_{xy} = \sin\theta$ — sine Gordon eq.

- In general

$\theta(x, y)$ satisfies (*) $\theta_{xy} = \sin\theta$.

$$\Rightarrow \tilde{\theta}(x, y) := \theta\left(\mu x, \frac{y}{\mu}\right)$$

is also a sol. of sine Gordon

"spectral parameter"

Exercise 3-2

• "uniqueness" of the asymptotic Chebyshev net

Problem

Let (ξ, η) be an asymptotic Chebyshev net on a surface. Assume another parameter (x, y) is also an asymptotic Chebyshev net. Prove that (x, y) satisfies

$$(x, y) = (\pm\xi + x_0, \pm\eta + y_0) \quad \text{or} \quad (x, y) = (\pm\eta + x_0, \pm\xi + y_0)$$

where x_0 and y_0 are constants.

$$\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \neq 0$$

$$\begin{cases} x_\xi y_\xi = 0 \\ x_\eta y_\eta = 0 \end{cases}$$

$x_\xi = \pm 1$

$y_\eta = \pm 1$

- $ds^2 = d\xi^2 + 2\cos\theta\, d\xi\, d\eta + d\eta^2$.
  $= dx^2 + 2\cos\varphi\, dx\, dy + dy^2$.
- $II = 2\cancel{\cos}\,\theta\, d\xi\, d\eta$.
  $= 2\cancel{\cos}\,\varphi\, dx\, dy$.

$\sin$

- $\boxed{y_\eta \neq 0}$

$x_\eta = 0 \Rightarrow x = x(\xi)$

$\dot{x} \neq 0$

$\Rightarrow y_\xi = 0$

$\Rightarrow y = y(\eta)$

$II = 2\sin\varphi\, dx\, dy$

$= 2\sin\varphi\,(x_\xi\, d\xi + x_\eta\, d\eta)(y_\xi\, d\xi + y_\eta\, d\eta)$

$= 2\sin\varphi\,\{(x_\xi y_\xi)\, d\xi^2 + (\quad)\, d\xi\, d\eta + (x_\eta y_\eta)\, d\eta^2\}$