Advanced Topics in Geometry B1 (MTH.B406)

A construction of pseudospherical surfaces

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Exercise 3-1

Problem

Let a and b be real numbers with $a \neq 0$ and

$$p(u, v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0, 0, u).$$

Find a coordinate change $(u, v) \mapsto (\xi, \eta)$ to an asymptotic Chebyshev net for p, and give an explicit expression of θ as a function in (ξ, η).

Exercise 3-1

$\rho := \sqrt{a^2 + b^2} > 0$

$$\begin{aligned} ds^2 &= (a^2 \operatorname{sech}^2 v + b^2)\, du^2 + 2ab \tanh^2 v\, du\, dv + a^2 \tanh^2 v\, dv^2 \\ &= \rho^2\, du^2 - a^2 \tanh^2 v(a^2\, du^2 - 2ab\, du\, dv - a^2\, dv^2) \\ II &= -\rho^{-1} \operatorname{sech} v \tanh v(a^2\, du^2 - 2ab\, du\, dv - a^2\, dv^2) \end{aligned}$$

Exercise 3-1

$$
\begin{aligned}
a^2\,du^2 - 2ab\,du\,dv - a^2\,dv^2 &= (a\,du - b\,dv)^2 - (\rho\,dv)^2 \\
&= (a\,du - (b+\rho)\,dv)(a\,du - (b-\rho)\,dv) \\
&= d\xi\,d\eta
\end{aligned}
$$

where

$$\xi = au - (b+\rho)v, \qquad \eta = au - (b-\rho)v.$$

Exercise 3-1

$$
\begin{aligned}
ds^2 &= \rho^2\, du^2 - a^2 \tanh^2 v\, d\xi\, d\eta \\
II &= -\rho^{-1} \operatorname{sech} v \tanh v\, dx\, d\eta \\
\begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \begin{pmatrix} a & -(b+\rho) \\ a & -(b-\rho) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \\
\begin{pmatrix} u \\ v \end{pmatrix} &= \frac{1}{2a\rho} \begin{pmatrix} -(b-\rho) & (b+\rho) \\ -a & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.
\end{aligned}
$$

$$
ds^2 = \left(\frac{b-\rho}{2a} d\xi\right)^2 + \left(\frac{1}{2} - \tanh^2 v\right) d\xi\, d\eta + \left(\frac{b+\rho}{2a} d\eta\right)^2
$$

Exercise 3-1

$$x = \frac{b-\rho}{2a}\xi, \qquad y = \frac{b+\rho}{2a}\eta \qquad \Rightarrow$$

$$\begin{aligned} ds^2 &= dx^2 + 2(2\tanh^2 v - 1)\,dx\,dy + dy^2 \\ &= dx^2 + 2\cos\theta\,dx\,dy + dy^2 \\ II &= 4\rho^{-1}\operatorname{sech} v \tanh v\,dx\,dy \\ &= 2\rho^{-1}\sin\theta\,dx\,dy \end{aligned}$$

Set

$$\cos\frac{\theta}{2} = \tanh v, \qquad \sin\frac{\theta}{2} = \operatorname{sech} v,$$

Exercise 3-1

$$\cos\frac{\theta}{2} = \tanh v, \quad \sin\frac{\theta}{2} = \operatorname{sech} v, \qquad v = \frac{1}{2\rho}\left(\frac{2a}{\rho - b}x + \frac{2a}{\rho + b}y\right)$$

Exercise 3-2

Problem

Let (ξ, η) be an asymptotic Chebyshev net on a surface. Assume another parameter (x, y) is also an asymptotic Chebyshev net. Prove that (x, y) satisfies

$$(x, y) = (\pm\xi + x_0, \pm\eta + y_0) \quad \text{or} \quad (x, y) = (\pm\eta + x_0, \pm\xi + y_0)$$

where x_0 and y_0 are constants.

Exercise 3-2

$$
\begin{aligned}
ds^2 &= d\xi^2 + 2\cos\theta\, d\xi\, d\eta + d\eta^2 \\
&= dx^2 + 2\cos\varphi\, dx\, dy + dy^2 \\
II &= 2\cos\theta\, d\xi\, d\eta \\
&= 2\cos\varphi\, dx\, dy
\end{aligned}
$$