• ∃ asymptotic Chebyshev net ← pseudospherical surface.

# Advanced Topics in Geometry B1 (MTH.B406)

A construction of pseudospherical surfaces
(general)

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Exercise 2-1 : surfaces of revolution.

## Today's Goal

Fundamental Theorem for surface theory

### Theorem

*Let $\theta\colon U \to (0,\pi)$ be a smooth function defined on a simply connected domain $U \subset \mathbb{R}^2$ satisfying the sine-Gordon equation*

$$\theta_{xy} = \sin\theta .$$

← "Gauss eq"

*Then there exists a regular parametrization $\underline{p\colon U \to \mathbb{R}^3}$ of a pseudospherical surface whose first and second fundamental forms are written as*

$$ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2, \qquad II = 2\sin\theta\, dx\, dy.$$

Codazzi

Chebyshev net.

→ explicit construction.

# Coordinate Change

$p\colon U \to \mathbb{R}^3$: a pseudospherical surface ($K = -1$):

$$ds^2 = dx^2 + 2\cos\theta\,dx\,dy + dy^2, \qquad II = 2\sin\theta\,dx\,dy$$

$$x = \frac{1}{2}(u - v), \qquad y = \frac{1}{2}(u + v)$$

$\Rightarrow$

$$ds^2 = \cos^2\frac{\theta}{2}\,du^2 + \sin^2\frac{\theta}{2}\,dv^2, \qquad II = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2)$$

$$ds^2 = \frac{1}{4}(du - dv)^2 + 2\cos\theta\,\frac{1}{4}(du - dv)(du + dv) + \frac{1}{4}(du + dv)^2$$

$$= \left(\frac{1}{2} + \frac{1}{2}\cos\theta\right)du^2 + \left(\frac{1}{2} - \frac{1}{2}\cos\theta\right)dv^2$$

## Orthonormal frame

$\nu$: the unit normal
$\cdot$ : inner product.

$$ds^2 = \cos^2\frac{\theta}{2}\,du^2 + \sin^2\frac{\theta}{2}\,dv^2, \qquad II = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2)$$

$$p_u \cdot p_u = \cos^2\frac{\theta}{2}, \quad p_u \cdot p_v = 0, \quad p_v \cdot p_v = \sin^2\frac{\theta}{2}$$

$$p_{uu} \cdot \nu = \cos\frac{\theta}{2}\sin\frac{\theta}{2} = -p_{vv} \cdot \nu, \quad p_{uv} \cdot \nu = 0$$

Orthonormal frame:

$$\mathcal{F} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \underset{\underset{\nu}{\parallel}}{\boldsymbol{e}_3}), \qquad p_u = \cos\frac{\theta}{2}\boldsymbol{e}_1, \quad p_v = \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad \nu = \boldsymbol{e}_3$$

# The Gauss-Weingarten equation

$(e_1 \; e_2 \; e_3)_u = (e_1 \; e_2 \; e_3)\,\Omega$

skew symmetric

$$\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & -\theta_v/2 & -\sin\frac{\theta}{2} \\ \theta_v/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix},$$

$$\mathcal{F}_v = \mathcal{F}\Lambda = \mathcal{F}\begin{pmatrix} 0 & -\theta_u/2 & 0 \\ \theta_u/2 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

$$\boxed{\theta_{uu} - \theta_{vv} = \sin\theta}$$

$p_{uu} \cdot e_3 = \cancel{\cos\frac{\theta}{2}} (e_1)_u \cdot e_3$

$\parallel$

$p_{uu} \cdot \nu$

$\parallel$

$\sin\frac{\theta}{2} \cancel{\cos\frac{\theta}{2}}$

$p_{uu} = \left(\cos\frac{\theta}{2} e_1\right)_u = * \, e_1 + \cos\frac{\theta}{2} (e_1)_u$

$\cancel{\sin\frac{\theta}{2}} \cos\frac{\theta}{2} \cdot (e_1)_u \cdot e_2 = \sin\frac{\theta}{2} \, p_{uu} \cdot e_2 = p_{uu} \cdot p_v$

$= (p_u \cdot p_v)_u - (p_u \cdot p_{uv})$

$= -\frac{1}{2}(p_u \cdot p_u)_v$

$\sin^2\frac{\theta}{2}$

$= \frac{\theta_u}{2} \sin\cancel{\frac{\theta}{2}} \cos\cancel{\frac{\theta}{2}}$

# The Fundamental Theorem

**Theorem**

*Let $\theta \colon U \to (0, \pi)$ be a smooth function defined on a simply connected domain $U \subset \mathbb{R}^2$ satisfying the sine-Gordon equation*

$$\text{Solve this } \theta_{uu} - \theta_{vv} = \sin\theta \quad \to \quad \mathcal{F} = (e_1 \ e_2 \ e_3)$$

*Then there exists a regular parametrization $p \colon U \to \mathbb{R}^3$ of a pseudospherical surface whose first and second fundamental forms are written as*

$$ds^2 = \cos^2\frac{\theta}{2}\,du^2 + \sin^2\frac{\theta}{2}\,dv^2, \quad II = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2)$$

$$dp = p_u\,du + p_v\,dv = \cos\frac{\theta}{2}\,e_1\,du + \sin\frac{\theta}{2}\,e_2\,dv$$

# Example

Assume

$\theta = \theta(v)$:

$$\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & -\dot{\theta}/2 & -\sin\frac{\theta}{2} \\ \dot{\theta}/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix};$$

$$\mathcal{F}_v = \mathcal{F}\Lambda = \mathcal{F}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

$$\boxed{\ddot{\theta} = -\sin\theta} \leftarrow \text{eq. for pendulum}$$

## Solving Sine-Gordon equation

$$\frac{d}{dt}\left(\frac{1}{2}(\dot\theta)^2 - \cos\theta\right) = 0$$

1st integral

$$\ddot\theta = -\sin\theta \qquad \Rightarrow \qquad \frac{1}{\cancel{2}\;4}\dot\theta^2 + \sin^2\frac{\theta}{2} = E^2 \qquad (E > 0)$$

$$c := c(v) = \frac{\dot\theta}{2E}, \qquad s := s(v) = \frac{1}{E}\sin\frac{\theta}{2}$$

$$\Rightarrow \qquad c^2 + s^2 = 1, \quad (\dot c, \dot s) = \cos\frac{\theta}{2}(-s, c)$$

$$\dot c = \frac{\ddot\theta}{2E} = \frac{-\sin\theta}{2E} = \frac{-\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{E}$$

$$\frac{1}{2}\dot\theta^2 - \cos\theta = \frac{1}{2}\dot\theta^2 + 1 - \cos\theta - 1$$

$$= 2\left(\frac{1}{4}\dot\theta^2 + \frac{1-\cos\theta}{2}\right) - 1$$

$$\sin^2\frac{\theta}{2}$$

$$E\begin{pmatrix} 0 & -c & -s \\ c & 0 & 0 \\ s & 0 & 0 \end{pmatrix}$$

(v) $\qquad \Omega$

$$\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & -\dot\theta/2 & -\sin\frac{\theta}{2} \\ \dot\theta/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix}$$

0 - eigenvector

$$\Rightarrow \quad (\mathcal{F}P)_u = (\mathcal{F}P)\begin{pmatrix} 0 & -E & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\mathcal{F} = F_0(v)R(u)P^T(v)} \qquad R(u) = \begin{pmatrix} \cos Eu & -\sin Eu & 0 \\ \sin Eu & \cos Eu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{F}P = F_0(v)R(u)$$

$$P^T\Omega P = \begin{pmatrix} 0 & -E & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

$$\mathcal{F} = F_0(v) R(u) P^T(v)$$

$$\underline{\mathcal{F}_v = \mathcal{F}\Lambda}$$

$F_0$: cnst

$$\Rightarrow \quad \underline{\dot{F}_0 = O}$$

$$\Rightarrow \quad \underline{\mathcal{F} = R(u) P^T(v)}$$

$$\Rightarrow \boldsymbol{e}_1 = \boldsymbol{u}_1, \quad \boldsymbol{e}_2 = c(v)\boldsymbol{u}_2 + s(v)\boldsymbol{u}_3$$

$$R(u) = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) \begin{pmatrix} \cos Eu & -\sin Eu & 0 \\ \sin Eu & \cos Eu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Finding the parametrization

$$\boldsymbol{e}_1 = \boldsymbol{u}_1, \quad \boldsymbol{e}_2 = c(v)\boldsymbol{u}_2 + s(v)\boldsymbol{u}_3.$$

$$(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \begin{pmatrix} \cos Eu & -\sin Eu & 0 \\ \sin Eu & \cos Eu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; c(v) = \frac{\dot\theta}{2E}, \; s(v) = \frac{1}{E}\sin\frac{\theta}{2}$$

$$dp = p_u\,du + p_v\,dv = \cos\frac{\theta}{2}\boldsymbol{e}_1\,du + \sin\frac{\theta}{2}\boldsymbol{e}_2\,dv$$

$$= \cos\frac{\theta}{2}\, u_1\, du + \Big( \underbrace{\sin\frac{\theta}{2}\frac{\dot\theta}{2E}}\, u_2 + \frac{1}{E}\sin\frac{\theta}{2}\, u_3 \Big) dv$$

$$= \cos\frac{\theta}{2}\,\frac{1}{E}\underbrace{\begin{pmatrix} -\sin Eu \\ \sin Eu \\ 0 \end{pmatrix}}_{u} du + \frac{u_2}{E}\underbrace{\Big(-\cos\frac{\theta}{2}\Big)_v dv} + \underbrace{\frac{1}{E}\sin\frac{\theta}{2}}\,dv\, u_3$$

# Result

$$\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ 0 \end{pmatrix}$$

$$p = \frac{-2}{E}\cos\frac{\theta}{2}v_2 + \frac{1}{E}v_3\int_{v_0}^{v}\sin\frac{\theta(t)}{2}\,dt,$$

$$\mathfrak{n}_2 \qquad \mathfrak{n}_3 \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

p: a surface of revolution.

Q When $E=1$, the surface is congruent to the pseudosphere.

# Exercise 4-1

$\theta_{uu} - \theta_{vv} = \sin\theta$

## Problem

*The constant function* $\theta(u,v) = 0$ *is a solution of the sine-Gordon equation* $\theta_{uu} - \theta_{vv} = \sin\theta$ *although it does not satisfy the condition* $0 < \theta < \pi$*. In this case, explain what happens on the solution of the Gauss-Weingarten equation and resulting "surface"* $p(u,v)$*.*

# Exercise 4-2

Let $\theta = \theta(x, y)$ be a solution of the sine-Gordon equation $\theta_{xy} = \sin\theta$. Assume a function $\varphi$ satisfies

$$\left(\frac{\varphi - \theta}{2}\right)_x = a \sin\frac{\varphi + \theta}{2}, \qquad \left(\frac{\varphi + \theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi - \theta}{2},$$

where $a$ is a non-zero constant. Prove that $\varphi$ is also a solution of the sine-Gordon equation.

$\varphi_{xy} = \sin\varphi$

Bäcklund transformation.

"Bäcklund's theorem" (transformation of pseudospherical surface)