Advanced Topics in Geometry B1 (MTH.B406)

A construction of pseudospherical surfaces

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Today's Goal

Theorem

Let $\theta\colon U \to (0,\pi)$ *be a smooth function defined on a simply connected domain* $U \subset \mathbb{R}^2$ *satisfying the sine-Gordon equation*

$$\theta_{xy} = \sin\theta$$

Then there exists a regular parametrization $p\colon U \to \mathbb{R}^3$ *of a pseudospherical surface whose first and second fundamental forms are written as*

$$ds^2 = dx^2 + 2\cos\theta\,dx\,dy + dy^2, \qquad II = 2\sin\theta\,dx\,dy.$$

$\to$ explicit construction.

Coordinate Change

$p\colon U \to \mathbb{R}^3$: a pseudospherical surface ($K=-1$):

$$ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2, \qquad II = 2\sin\theta\, dx\, dy$$

$$x = \frac{1}{2}(u-v), \qquad y = \frac{1}{2}(u+v)$$

$\Rightarrow$

$$ds^2 = \cos^2\frac{\theta}{2}\, du^2 + \sin^2\frac{\theta}{2}\, dv^2, \qquad II = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2)$$

Orthonormal frame

$$ds^2 = \cos^2\frac{\theta}{2}\,du^2 + \sin^2\frac{\theta}{2}\,dv^2, \qquad II = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2)$$

$$p_u \cdot p_u = \cos^2\frac{\theta}{2}, \quad p_u \cdot p_v = 0, \quad p_v \cdot p_v = \sin^2\frac{\theta}{2}$$

$$p_{uu} \cdot \nu = \cos\frac{\theta}{2}\sin\frac{\theta}{2} = -p_{vv}\cdot\nu, \quad p_{uv}\cdot\nu = 0$$

Orthonormal frame:

$$\mathcal{F} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3), \qquad p_u = \cos\frac{\theta}{2}\boldsymbol{e}_1, \quad p_v = \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad \nu = \boldsymbol{e}_3$$

The Gauss-Weingarten equation

$$\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & -\theta_v/2 & -\sin\frac{\theta}{2} \\ \theta_v/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix},$$

$$\mathcal{F}_v = \mathcal{F}\Lambda = \mathcal{F}\begin{pmatrix} 0 & -\theta_u/2 & 0 \\ \theta_u/2 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

$$\boxed{\theta_{uu} - \theta_{vv} = \sin\theta}$$

The Fundamental Theorem

Theorem

Let $\theta\colon U \to (0,\pi)$ *be a smooth function defined on a simply connected domain* $U \subset \mathbb{R}^2$ *satisfying the sine-Gordon equation*

$$\theta_{uu} - \theta_{vv} = \sin\theta$$

Then there exists a regular parametrization $p\colon U \to \mathbb{R}^3$ *of a pseudospherical surface whose first and second fundamental forms are written as*

$$ds^2 = \cos^2\frac{\theta}{2}\,du^2 + \sin^2\frac{\theta}{2}\,dv^2, \quad II = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2)$$

Example

$\theta = \theta(v)$:

$$\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & -\dot{\theta}/2 & -\sin\frac{\theta}{2} \\ \dot{\theta}/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix},$$

$$\mathcal{F}_v = \mathcal{F}\Lambda = \mathcal{F}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

$$\boxed{\ddot{\theta} = -\sin\theta}$$

Solving Sine-Gordon equation

$$\ddot{\theta} = -\sin\theta \qquad \Rightarrow \qquad \frac{1}{2}\dot{\theta}^2 + \sin^2\frac{\theta}{2} = E^2 \qquad (E > 0)$$

$$c := c(v) = \frac{\dot{\theta}}{2E}, \qquad s := s(v) = \frac{1}{E}\sin\frac{\theta}{2}$$

$$\Rightarrow \qquad c^2 + s^2 = 1, \quad (\dot{c}, \dot{s}) = \cos\frac{\theta}{2}(-s, c)$$

Solving Gauss-Weingarten equation

$$\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & -\dot\theta/2 & -\sin\frac{\theta}{2} \\ \dot\theta/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad (\mathcal{F}P)_u = (\mathcal{F}P)\begin{pmatrix} 0 & -E & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{F} = F_0(v)R(u)P^T(v) \qquad R(u) = \begin{pmatrix} \cos Eu & -\sin Eu & 0 \\ \sin Eu & \cos Eu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solving Gauss-Weingarten equation

$\mathcal{F} = F_0(v)R(u)P^T(v)$

$$
\begin{aligned}
\mathcal{F}_v &= \mathcal{F}\Lambda \\
&\Rightarrow \quad \dot{F}_0 = O \\
&\Rightarrow \quad \mathcal{F} = R(u)P^T(v) \\
&\Rightarrow \boldsymbol{e}_1 = \boldsymbol{u}_1, \quad \boldsymbol{e}_2 = c(v)\boldsymbol{u}_2 + s(v)\boldsymbol{u}_3
\end{aligned}
$$

$$
R(u) = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)\begin{pmatrix} \cos Eu & -\sin Eu & 0 \\ \sin Eu & \cos Eu & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Finding the parametrization

$$\boldsymbol{e}_1 = \boldsymbol{u}_1, \quad \boldsymbol{e}_2 = c(v)\boldsymbol{u}_2 + s(v)\boldsymbol{u}_3$$

$$(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \begin{pmatrix} \cos Eu & -\sin Eu & 0 \\ \sin Eu & \cos Eu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; c(v) = \frac{\dot{\theta}}{2E}, \; s(v) = \frac{1}{E}\sin\frac{\theta}{2}$$

$$dp = p_u\,du + p_v\,dv = \cos\frac{\theta}{2}\boldsymbol{e}_1\,du + \sin\frac{\theta}{2}\boldsymbol{e}_2\,dv$$

Result

$$p = \frac{-2}{E}\cos\frac{\theta}{2}\boldsymbol{v}_2 + \frac{1}{E}\boldsymbol{v}_3\int_{v_0}^{v}\sin\frac{\theta(t)}{2}\,dt,$$

Exercise 4-1

Problem

The constant function $\theta(u,v) = 0$ *is a solution of the sine-Gordon equation* $\theta_{uu} - \theta_{vv} = \sin\theta$ *although it does not satisfy the condition* $0 < \theta < \pi$*. In this case, explain what happens on the solution of the Gauss-Weingarten equation and resulting "surface"* $p(u,v)$*.*

Exercise 4-2

Let $\theta = \theta(x, y)$ be a solution of the sine-Gordon equation $\theta_{xy} = \sin\theta$. Assume a function φ satisfies

$$\left(\frac{\varphi - \theta}{2}\right)_x = a \sin\frac{\varphi + \theta}{2}, \qquad \left(\frac{\varphi + \theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi - \theta}{2},$$

where a is a non-zero constant. Prove that φ is also a solution of the sine-Gordon equation.