Advanced Topics in Geometry B1 (MTH.B406)

Hilbert's theorem

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Exercise 4-1

$\boxed{0 < \theta < \pi}$ regularity.

Problem

The constant function $\theta(u, v) = 0$ *is a solution of the sine-Gordon equation* $\theta_{uu} - \theta_{vv} = \sin\theta$ *although it does not satisfy the condition* $0 < \theta < \pi$*. In this case, explain what happens on the solution of the Gauss-Weingarten equation and resulting "surface"* $p(u, v)$*.*

The Gauss-Weingarten equation

$p_u = \overset{1}{\cos\frac{\theta}{2}}\, e_1$

$p_v = \overset{0}{\sin\frac{\theta}{2}}\, e_2$

$$\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & \cancel{-\theta_v/2}\ 0 & \cancel{-\sin\frac{\theta}{2}}\ 0 \\ \cancel{\theta_v/2}\ 0 & 0 & 0 \\ \cancel{\sin\frac{\theta}{2}}\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{F} = \mathcal{F}(v)$$

$$\mathcal{F}_v = \mathcal{F}\Lambda = \mathcal{F}\begin{pmatrix} 0 & \cancel{-\theta_u/2} & 0 \\ \cancel{\theta_u/2}\ 0 & 0 & \overset{1}{\cos\frac{\theta}{2}} \\ 0 & \underset{-1}{-\cos\frac{\theta}{2}} & 0 \end{pmatrix}.$$

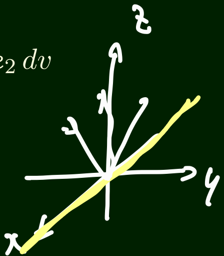
$$\boxed{\theta_{uu} - \theta_{vv} = \sin\theta}$$

$$\mathcal{F}' = \mathcal{F}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \mathcal{F} = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & \sin v \\ 0 & -\sin v & \cos v \end{pmatrix} \end{array}$$

Corresponding “surface”

$$dp = p_u\,du + p_v\,dv = \overset{1}{\cos\frac{\theta}{2}}\boldsymbol{e}_1\,du + \overset{0}{\sin\frac{\theta}{2}}\boldsymbol{e}_2\,dv$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} du$$



- $p(u,v) = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}$ straight line.
  - ∀ points are singular
- $\nu(u,v) = \begin{pmatrix} 0 \\ \sin v \\ \cos v \end{pmatrix}$

★ $\theta_{xy} = \sin\theta$

Let $\theta = \theta(x, y)$ be a solution of the sine-Gordon equation $\theta_{xy} = \sin\theta$. Assume a function φ satisfies

$$① \left(\frac{\varphi - \theta}{2}\right)_x = a \sin\frac{\varphi + \theta}{2}, \quad ② \left(\frac{\varphi + \theta}{2}\right)_y = \frac{1}{a} \sin\frac{\varphi - \theta}{2},$$

where a is a non-zero constant. Prove that φ is also a solution of the sine-Gordon equation.

$\varphi_{xy} = \sin\varphi \xleftarrow{\text{easy}} (①)_y = (②)_x$

$\theta_{xy} = \sin\theta$

$\left.\begin{array}{l}\theta: \text{known} \\ \varphi: \text{unkown}\end{array}\right) \Rightarrow$ the compatibility : ?

Q and A

Q: For Exercise 4-2, I think it is enough if θ is of class C^2 (or C^∞), but you assume $\theta_{xy} = \sin\theta$. Is it because you want to emphasize that we can find a new φ from θ that satisfies the sine-Gordon equation?

Exercise 4-2

θ: sol. of sine Gordon

$\Rightarrow$ so is φ

$\theta \rightsquigarrow \varphi$

- $$\left(\frac{\varphi-\theta}{2}\right)_x = a\sin\frac{\varphi+\theta}{2}, \qquad \left(\frac{\varphi+\theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi-\theta}{2},$$

example $\theta = 0$ (vacuum)

$$\left(\frac{\varphi}{2}\right)_x = a\sin\frac{\varphi}{2} \qquad \left(\frac{\varphi}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi}{2}$$

$\rightsquigarrow$ $\underline{\varphi(x,y)}$

$\rightsquigarrow$ pseudospherical surface

- pseudosphere if $a = 1$
- Dini's pseudosphere

Bäcklund transformation

- A transformation of solutions of the sine-Gordon equation.

  "integrable system"

- A transformation of surfaces.

  pseudospherical

Bäcklund's theorem

Definition

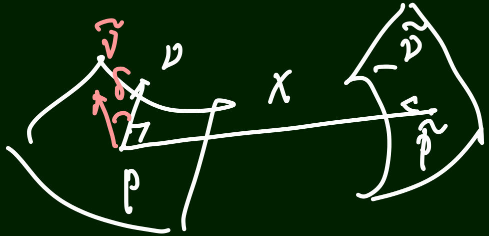
Let $p(u, v)$ be a parametrization of a surface. A parametrized surface $\tilde{p}(u, v)$ is a Bäcklund transformation of p if it satisfies

- $|X| = r \neq 0$ is constant,
- $X(u, v)$ tangent to the surface p at $p(u, v)$,
- $X(u, v)$ tangent to the surface $\tilde{p}$ at $\tilde{p}(u, v)$,
- the angle δ of the unit normal ν of p and $\tilde{\nu}$ of $\tilde{p}$ is constant,

where $X := \tilde{p} - p$.

Theorem (Bäcklund) 19c

If a Bäcklund transformation of a surface exists, the Gaussian curvature K the surface is constant $K = -\sin^2 \delta / r^2$. Conversely, a surface of constant negative Gaussian curvature admits a Bäcklund transformation.



$\tilde{v}$
v
x
p
$\tilde{v}$
$\tilde{p}$

Example

When $\theta = 0$...

$$\left(\frac{\varphi-\theta}{2}\right)_x = a\sin\frac{\varphi+\theta}{2}, \qquad \left(\frac{\varphi+\theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi-\theta}{2},$$

$\to$ pseudosphere (1-soliton)

$\exists_\infty$ pseudospherical surfaces

~~seem~~ to have singularities