# Advanced Topics in Geometry B1 (MTH.B406)

## Hilbert's theorem

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# Exercise 4-1

## Problem

*The constant function* $\theta(u, v) = 0$ *is a solution of the sine-Gordon equation* $\theta_{uu} - \theta_{vv} = \sin\theta$ *although it does not satisfy the condition* $0 < \theta < \pi$. *In this case, explain what happens on the solution of the Gauss-Weingarten equation and resulting "surface"* $p(u, v)$.

# The Gauss-Weingarten equation

$$
\mathcal{F}_u = \mathcal{F}\Omega = \mathcal{F}\begin{pmatrix} 0 & -\theta_v/2 & -\sin\frac{\theta}{2} \\ \theta_v/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix},
$$

$$
\mathcal{F}_v = \mathcal{F}\Lambda = \mathcal{F}\begin{pmatrix} 0 & -\theta_u/2 & 0 \\ \theta_u/2 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}.
$$

$$
\boxed{\theta_{uu} - \theta_{vv} = \sin\theta}
$$

# Corresponding “surface”

$$dp = p_u\,du + p_v\,dv = \cos\frac{\theta}{2}\boldsymbol{e}_1\,du + \sin\frac{\theta}{2}\boldsymbol{e}_2\,dv$$

# Exercise 4-2

Let $\theta = \theta(x, y)$ be a solution of the sine-Gordon equation $\theta_{xy} = \sin\theta$. Assume a function $\varphi$ satisfies

$$\left(\frac{\varphi - \theta}{2}\right)_x = a \sin\frac{\varphi + \theta}{2}, \qquad \left(\frac{\varphi + \theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi - \theta}{2},$$

where $a$ is a non-zero constant. Prove that $\varphi$ is also a solution of the sine-Gordon equation.

# Q and A

Q: For Exercise 4-2, I think it is enough if $\theta$ is of class $C^2$ (or $C^\infty$), but you assume $\theta_{xy} = \sin\theta$. Is it because you want to emphasize that we can find a new $\varphi$ from $\theta$ that satisfies the sine-Gordon equation?

# Exercise 4-2

$$\left(\frac{\varphi-\theta}{2}\right)_x = a\sin\frac{\varphi+\theta}{2}, \qquad \left(\frac{\varphi+\theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi-\theta}{2},$$

# Bäcklund transformation

- A transformation of solutions of the sine-Gordon equation.
- A transformation of surfaces.

# Bäcklund's theorem

## Definition

Let $p(u,v)$ be a parametrization of a surface. A parametrized surface $\tilde{p}(u,v)$ is a Bäcklund transformation of $p$ if it satisfies

- $|X| = r \neq 0$ is constant,
- $X(u,v)$ tangent to the surface $p$ at $p(u,v)$,
- $X(u,v)$ tangent to the surface $\tilde{p}$ at $\tilde{p}(u,v)$,
- the angle $\delta$ of the unit normal $\nu$ of $p$ and $\tilde{\nu}$ of $\tilde{p}$ is constant,

where $X := \tilde{p} - p$.

## Theorem (Bäcklund)

*If a Bäcklund transformation of a surface exists, the Gaussian curvature $K$ the surface is constant $K = -\sin^2\delta/r^2$. Conversely, a surface of constant negative Gaussian curvature admits a Bäcklund transformation.*

# Example

When $\theta = 0$...

$$\left(\frac{\varphi - \theta}{2}\right)_x = a \sin \frac{\varphi + \theta}{2}, \qquad \left(\frac{\varphi + \theta}{2}\right)_y = \frac{1}{a} \sin \frac{\varphi - \theta}{2},$$