Advanced Topics in Geometry B1 (MTH.B406)

Hilbert's theorem

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2025/07/11

Today's Goal

Theorem (Hilbert, 1901)

There exists no complete pseudospherical surface.

a pseudospherical surface: $K = -1$.

Completeness

Riemannian metric

$(ds^2)_p$: inner product of T_pM.

Definition

A Riemannian manifold (M, ds^2) is complete if the induced distance function d_{ds^2} is complete.

- $ds^2 \rightsquigarrow d_{ds^2}$ distance function

$\forall$ Cauchy sequence w.r.to d_{ds^2} converges.

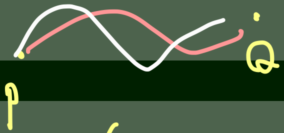
Completeness

- (M, ds^2): a Riemannian manifold;
- $\gamma\colon [a,b] \to M$: a curve.
- $\mathcal{C}_{\mathrm{P},\mathrm{Q}}$: the set of curves of M joining P and Q.

Definition (Length)

$$\mathcal{L}_{ds^2}(\gamma) := \int_a^b \underbrace{|\gamma'(t)|}_{\text{speed}}\, dt, \quad \text{where} \quad |\gamma'(t)| = \sqrt{ds^2(\gamma'(t), \gamma'(t))}.$$

Completeness



Definition (Distance)

$$d_{ds^2}(\mathrm{P},\mathrm{Q}) := \underline{\inf}\left\{\underline{\mathcal{L}_{ds^2}(\gamma)}\,;\,\gamma\in\mathcal{C}_{\mathrm{P},\mathrm{Q}}\right\},$$

- Fact: d_{ds^2} is a distance on M. omit

Definition

A Riemannian manifold (M, ds^2) is complete if the induced distance function d_{ds^2} is complete.

Example

$T_p\mathbb{R}^2 = (\mathbb{R}^2, \langle\,,\rangle)$

- $\mathbb{R}^2$: the Euclidean plane

$\gamma(t) = (x(t), y(t)) \qquad a \le t \le b$

$$\mathcal{L}_{ds^2}(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$\cdot\ d_{ds^2}(P, Q) = |\overrightarrow{PQ}| \leftarrow$ the shortest path joining P & Q is the line segment PQ

complete

- $\mathbb{R}^2 \setminus \{(0,0)\}$

incompete

$O \quad \left\{\left(\frac{1}{n}, 0\right)\right\}$: Cauchy.

does not converge

The hyperbolic plane

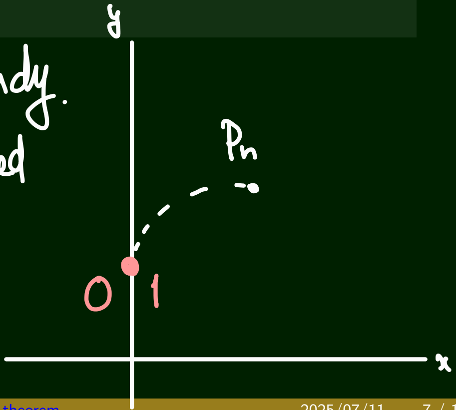
$$H^2 := \{(x, y)\,;\, y > 0\}, \qquad ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Proposition

(H^2, ds^2) *is complete.*

$\because$ $P_n = \{(x_n, y_n)\}$: Cauchy.

$\{d(O, P_n)\}$: bounded



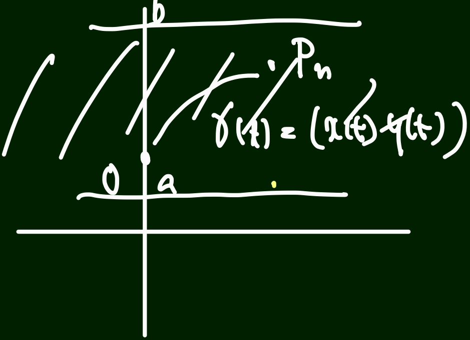
$$L(\gamma) = \int \frac{\sqrt{\cancel{\dot x^2} + \dot y^2}}{y}\, dt$$

inf

$$\geq \int \left| \frac{\dot y}{y} \right| dt$$

$$\geq \left| \int \frac{\dot y}{y}\, dt \right|$$

$$= \left| \log y_n \right| \qquad \therefore\ d(0, P_n) \geq |\log y_n|$$



$\Rightarrow |\log y_n|$: bounded $\Rightarrow 0 < \exists a < y_n < \exists b$

$$\frac{1}{b^2}\underbrace{(dx^2 + dy^2)}_{\text{usual metric}} \leq ds^2 = \frac{dx^2 + dy^2}{y^2} \leq \frac{1}{a^2}\underline{(dx^2 + dy^2)}$$

(x_n, y_n) is a Cauchy sequence
w.r.to the Euclidean metric $\Rightarrow$ converges

"'A Global realization of non-Euc. geom
is a pseudospherical surface."

complete.

Hilbert's theorem

Theorem (Hilbert, 1901)

There exists no complete pseudospherical surface.

($\nexists$ realization of non-Euclidean geom
as a surface in $\mathbb{R}^3$.)

Proof of Hilbert's theorem (Part 1)

- $p : M^2 \to \mathbb{R}^3$: complete immersion of constant Gaussian curvature -1.

Proposition (Global asymptotic Chebyshev net)

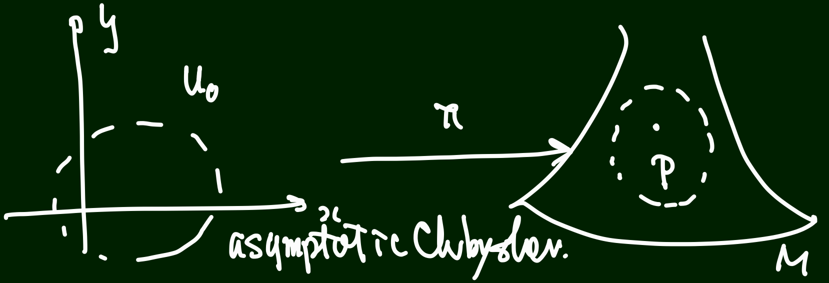
There exists a smooth map

$$\pi : \overset{(,)}{\mathbb{R}^2} \longrightarrow M \xrightarrow[p]{} \mathbb{R}^3$$

$(x,y) \to \mathbb{R}^3$

such that $\tilde{p} = \underline{p \circ \pi} : \mathbb{R}^2 \to \mathbb{R}^3$ *has first and second fundamental forms as*

$$ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2, \quad II = 2\sin\theta\, dx\, dy,$$
$$\underline{0 < \theta < \pi}, \quad \underline{\theta_{xy} = \sin\theta}$$



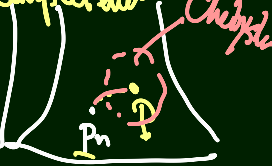
$\mathcal{U} = \{ U \overset{\text{open}}{\subset} \mathbb{R}^2 ;\ U_0 \subset U ,\ (x.y): \text{asymptotic Chebyshev} \}$ locally

U : maximal element of $\mathcal{U}$.

$U \subsetneq \mathbb{R}^2$

$\exists \{P_n\}$: Cauchy w.r.t. d_{ds^2}

Completeness



$\cdot\ \xi = \pm x + a\ ,\ \pm y + a$

$\cdot\ \eta = \pm y + b\ ,\ \pm x + b$

U: extended as asy Chebyshev net

contradiction.

⊛ $\{p_n\}$: cauchy . w.r.to euclidean metric.

$\cdot\ ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2 \leq 2(dx^2 + dy^2)$

$\Rightarrow \{p_n\}$: Cauchy w.r.to ds^2.

Proof of Hilbert's theorem (Part 2)

Proposition

There exists no smooth function $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ *such that*

- $\theta_{xy} = \sin\theta$
- $0 < \theta < \pi$.

$\Rightarrow$ Hilbert's Thm is proved.

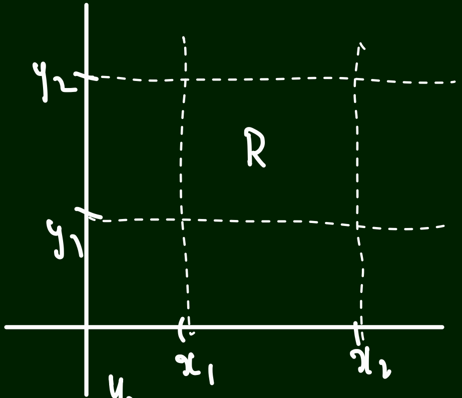
Proof of Hilbert's theorem (Part 2a)

- $\theta_{xy} = \sin\theta$,
- $x_1 < x_2$, $y_1 < y_2$

$0 < \theta < \pi$

Lemma

$$\theta(x_2, y_2) - \theta(x_1, y_2) = \theta(x_2, y_1) - \theta(x_1, y_1) + \int_{y_1}^{y_{\cancel{1}2}} dy \int_{x_1}^{x_2} dx \sin\theta(x, y).$$



$$\iint_R \sin\theta \, dx\, dy$$

$$= \iint_R \theta_{xy} \, dx\, dy$$

$$= \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} (\theta_y)_x \, dx$$

$$= \int_{y_1}^{y_2} dy \left(\theta_y(x_2, y) - \theta_y(x_1, y) \right)$$

$$= \theta(x_2\ y_2) - \theta(x_2\ y_1) - \theta(x_1\ y_2) + \theta(x_1\ y_1)$$

Proof of Hilbert's theorem (Part 2b)

by an appropriate coordinate change

- $\theta_{xy} = \sin\theta$,
- $x \mapsto \theta(x, 0)$ is strictly increasing on $[0, x_1]$

fix

if $x \mapsto \theta(x, y_1)$ is non const.

$\Rightarrow \exists$ an interval $[x_2, x_3]$: increasing/decr.

$\Rightarrow (x, y) \mapsto (-x, -y)$ increasing.

$\Rightarrow$ translation

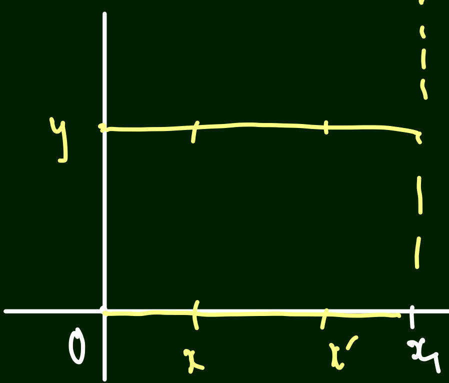
otherwise $\forall y_1$; $x \mapsto \theta(x, y_1)$ is constant 0

$\cancel{\theta(x_2, y_2) - \theta(x_1, y_2)} = \cancel{\theta(x_2, y_1) - \theta(x_1, y_1)} + \boxed{\iint \sin\theta \, dx\, dy}$

contradiction

Proof of Hilbert's theorem (Part 2c)

- $x \mapsto \theta(x, 0)$ is strictly increasing on $[0, x_1]$
- $x \mapsto \theta(x, y)$ is strictly increasing on $[0, x_1]$ for fixed $y > 0$.



$$\theta(x', y) - \theta(x, y)$$
$$= \theta(x', 0) - \theta(x, 0) + \iint \sin\theta \, dx\,dy$$
$$> \theta(x', 0) - \theta(x, 0)$$
$$> 0$$

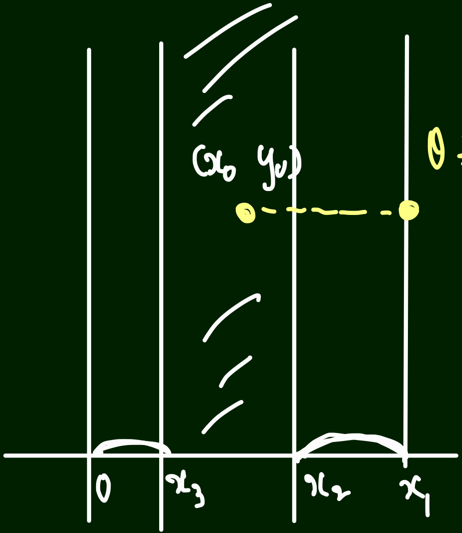
Proof of Hilbert's theorem (Part 2d)

- $0 < x_3 < x_2 < x_1$
- $\varepsilon := \theta(x_1, 0) - \theta(x_2, 0) > 0$
- $\varepsilon' := \theta(x_3, 0) - \theta(0, 0) > 0$

Lemma

There exists $(x_0, y_0) \in (x_3, x_2) \times (0, \infty)$ *such that*

$$\theta(x_0, y_0) > \pi - \frac{\varepsilon}{2}.$$



$\varepsilon = \theta(x_1, 0) - \theta(x_2, 0)$

$\varepsilon' = \theta(x_3, 0) - \theta(0, 0)$

$\theta \geq \pi - \frac{\varepsilon}{2}$ contradiction

$\exists\, (x_0, y_0)$

$$\boxed{\theta(x_0, y_0) \geq \pi - \frac{\varepsilon}{2}}$$

if $\theta(x, y) \leq \pi - \frac{\varepsilon}{2}$ on

$\theta(x.y) \geq \varepsilon'$

$\iint \sin\theta \, dx\, dy \to \infty$

$\boxed{\sin\theta \geq \delta > 0}$

Exercise 5-1

Problem

Consider a map

$$p\colon \mathbb{R}^2 \ni (u,v) \longmapsto (v\cosh u, v, v\sinh u) \in \mathbb{R}^3.$$

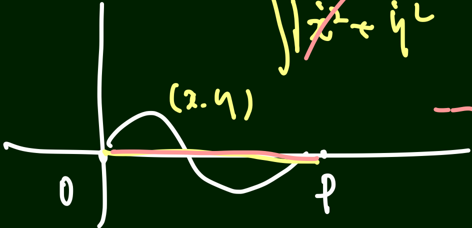
1. *Verify that the image* $p(\mathbb{R}^2)$ *is contained in the cone* $\{(x,y,z) \in \mathbb{R}^3\,;\, x^2 - y^2 - z^2 = 0\}$.
2. *Is the induced metric* $p^*\langle\ ,\ \rangle$ *complete on* $\mathbb{R}^2$*?*

Exercise 5-2

Problem

Prove that the shortest curve (with respect to the canonical Riemannian metric) joining $\mathrm{O} := (0,0)$ *and* $\mathrm{P} := (L,0)$ $(L > 0)$ *on the Euclidean plane is the line segment joining them.*

$$\int \sqrt{\cancel{\dot{x}^2} + \dot{y}^2}\, dt$$



— equality.