Advanced Topics in Geometry B1 (MTH.B406)

Hilbert's theorem

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Today's Goal

Theorem (Hilbert, 1901)

There exists no complete pseudospherical surface.

a pseudospherical surface: $K = -1$.

Completeness

Definition

A Riemannian manifold (M, ds^2) is complete if the induced distance function d_{ds^2} is complete.

Completeness

- (M, ds^2): a Riemannian manifold;
- $\gamma\colon [a,b] \to M$: a curve.
- $\mathcal{C}_{\mathrm{P},\mathrm{Q}}$: the set of curves of M joining P and Q.

Definition (Length)

$$\mathcal{L}_{ds^2}(\gamma) := \int_a^b |\gamma'(t)|\,dt, \quad \text{where} \quad |\gamma'(t)| = \sqrt{ds^2(\gamma'(t), \gamma'(t))}.$$

Completeness

Definition (Distance)

$$d_{ds^2}(\mathrm{P}, \mathrm{Q}) := \inf\left\{\mathcal{L}_{ds^2}(\gamma)\,;\,\gamma \in \mathcal{C}_{\mathrm{P},\mathrm{Q}}\right\},$$

- Fact: d_{ds^2} is a distance on M.

Definition

A Riemannian manifold (M, ds^2) is complete if the induced distance function d_{ds^2} is complete.

Example

- $\mathbb{R}^2$: the Euclidean plane

- $\mathbb{R}^2 \setminus \{(0,0)\}$

The hyperbolic plane

$$H^2 := \{(x, y)\,;\, y > 0\}, \qquad ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Proposition

(H^2, ds^2) *is complete.*

Hilbert's theorem

Theorem (Hilbert, 1901)

There exists no complete pseudospherical surface.

Proof of Hilbert's theorem (Part 1)

- $p : M \to \mathbb{R}^3$: complete immersion of constant Gaussian curvature -1.

Proposition (Global asymptotic Chebyshev net)

There exists a smooth map

$$\pi\colon \mathbb{R}^2 \longrightarrow M$$

such that $\tilde{p} = p \circ \pi\colon \mathbb{R}^2 \to \mathbb{R}^3$ *has first and second fundamental forms as*

$$ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2, \quad II = 2\sin\theta\, dx\, dy,$$
$$0 < \theta < \pi, \quad \theta_{xy} = \sin\theta$$

Proof of Hilbert's theorem (Part 2)

Proposition

There exists no smooth function $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ *such that*

- $\theta_{xy} = \sin\theta$
- $0 < \theta < \pi$.

Proof of Hilbert's theorem (Part 2a)

- $\theta_{xy} = \sin\theta$,
- $x_1 < x_2$, $y_1 < y_2$

Lemma

$$\theta(x_2, y_2) - \theta(x_1, y_2) = \theta(x_2, y_1) - \theta(x_1, y_1) + \int_{y_1}^{y^2} dy \int_{x_1}^{x_2} dx \sin\theta(x, y).$$

Proof of Hilbert's theorem (Part 2b)

- $\theta_{xy} = \sin\theta$,
- $x \mapsto \theta(x, 0)$ is strictly increasing on $[0, x_1]$

Proof of Hilbert's theorem (Part 2c)

- $x \mapsto \theta(x, 0)$ is strictly increasing on $[0, x_1]$
- $x \mapsto \theta(x, y)$ is strictly increasing on $[0, x_1]$ for fixed $y > 0$.

Proof of Hilbert's theorem (Part 2d)

- $0 < x_3 < x_2 < x_1$
- $\varepsilon := \theta(x_1, 0) - \theta(x_2, 0) > 0$
- $\varepsilon' := \theta(x_3, 0) - \theta(0, 0) > 0$

Lemma

There exists $(x_0, y_0) \in (x_3, x_2) \times (0, \infty)$ *such that*

$$\theta(x_0, y_0) > \pi - \frac{\varepsilon}{2}.$$

Exercise 5-1

Problem

Consider a map

$$p\colon \mathbb{R}^2 \ni (u,v) \longmapsto (v\cosh u, v, v\sinh u) \in \mathbb{R}^3.$$

1. *Verify that the image* $p(\mathbb{R}^2)$ *is contained in the cone* $\{(x,y,z) \in \mathbb{R}^3\,;\, x^2 - y^2 - z^2 = 0\}$.
2. *Is the induced metric* $p^*\langle\ ,\ \rangle$ *complete on* $\mathbb{R}^2$*?*

Exercise 5-2

Problem

Prove that the shortest curve (with respect to the canonical Riemannian metric) joining $\mathrm{O} := (0,0)$ *and* $\mathrm{P} := (L,0)$ $(L > 0)$ *on the Euclidean plane is the line segment joining them.*