Advanced Topics in Geometry B1 (MTH.B406)

Lorentz-Minkowski space

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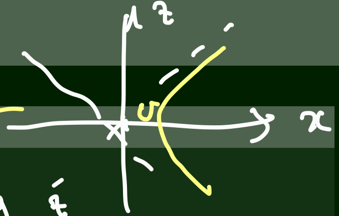
Exercise 5-1



Problem

Consider a map

$$p\colon \mathbb{R}^2 \ni (u,v) \longmapsto \underbrace{\overset{x}{(v\cosh u}, \overset{y}{v}, \overset{z}{v\sinh u)}}_{\text{cone}} \in \mathbb{R}^3.$$

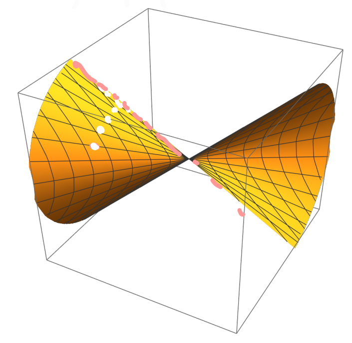
1. *Verify that the image* $p(\mathbb{R}^2)$ *is contained in the cone* $\{(x,y,z)\in\mathbb{R}^3\,;\, \underline{x^2-y^2-z^2=0}\}$.
2. *Is the induced metric* $p^*\langle\ ,\ \rangle$ *complete on* $\mathbb{R}^2$*?*

$y = v$: const (cross section of the image with the plane $y=v$)

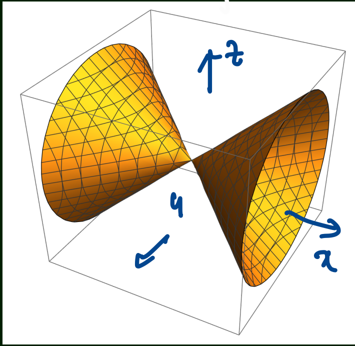
$$\begin{cases} x = v\cosh u \\ z = v\sinh u \end{cases}$$

Exercise 5-1

Image of P



$x^2-y^2-z^2=0$



Incomplete

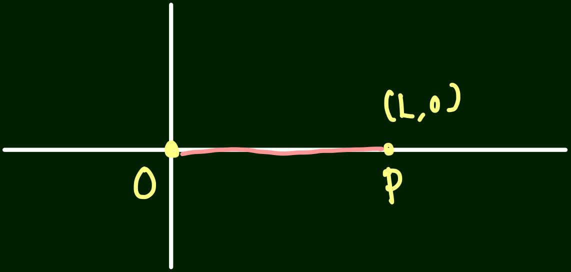
formulation : exercise

incompleteness : "lack of points"

Exercise 5-2

Problem

Prove that the shortest curve (with respect to the canonical Riemannian metric) joining $\mathrm{O} := (0, 0)$ *and* $\mathrm{P} := (L, 0)$ *(*$L > 0$*) on the Euclidean plane is the line segment joining them.*



Let $\underline{\gamma(t)} = (x(t), y(t))$ $0 \leq t \leq 1$ be a curve (C^1) joining O and P $\begin{cases} x(0) = 0 \\ y(0) = 0 \end{cases}$ $\begin{cases} x(1) = L > 0 \\ y(1) = 0 \end{cases}$

$$\underline{\mathcal{L}(\gamma)} = \int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt \underset{1}{\geq} \int_0^1 \sqrt{\dot{x}(t)^2}\, dt$$

$$= \int_0^1 |\dot{x}(t)|\, dt \underset{2}{\geq} \left| \int_0^1 \dot{x}(t)\, dt \right|$$

$$= | x(1) - x(0) | = \underline{L}$$

- the line segment joining O & P has length L.

  the line segment is a shortest path

- converse ?

$\mathcal{L}(\gamma) = L \Rightarrow$ " $\geq$ " should be " $=$ "

$\Rightarrow$ O_1 : $\dot{y}(t) = 0 \Rightarrow y$: const $\Rightarrow y = 0$

O_2 : $\dot{x}(t)$ does not change sign.

$(\dot{x} \geq 0)$

$\Rightarrow$ $\gamma(t) = (x(t), 0)$

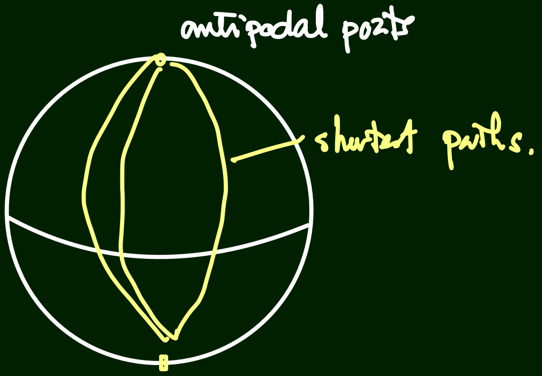
increasing (in wider sense)

non decreasing.

$\Rightarrow$ line segment.

This argument works for the hyperbolic plane.
(next lecture)

Almost works for the sphere.



Recall

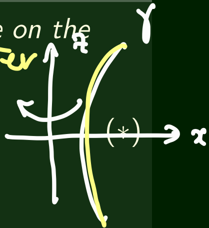
"pseudospherical surfaces of revolution."

Problem (Exercise 2-1)

Let $\gamma(t) = \big(x(t), z(t)\big)$ $(\gamma \in I)$ be a parametrized curve on the xz-plane satisfying

arclength parameter (unit speed)

$$\big(x'(t)\big)^2 + \big(z'(t)\big)^2 = 1 \qquad (t \in I), \qquad (*)$$



where $I \subset \mathbb{R}$ is an interval. Consider a surface

$$p_\gamma(s,t) := \big(x(t)\cos s, x(t)\sin s, z(t)\big),$$

which is a surface of revolution of profile curve γ.

1. *Show that p_γ is pseudospherical if and only if $\boxed{x'' = x}$ holds.*
2. *Can one choose $I = \mathbb{R}$?*

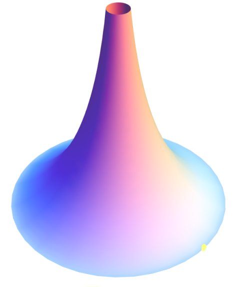
e^{-t}. $A \cosh t$. $A \sinh t$.

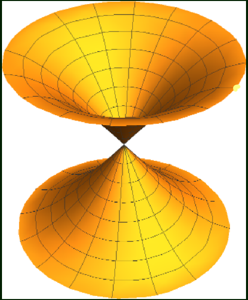
Exercise 2-1

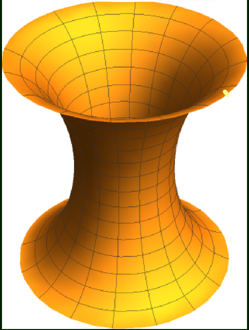
$x'^2 + z'^2 = 1 \leftarrow$ cannot hold whole on $\mathbb{R}$

Lorentzian geometry $\leftarrow$ $-$

$x = e^{-s}$ $\qquad$ $x = A\sinh s$ $\qquad$ $x = A\cosh s$







$|x'| < 1$