Advanced Topics in Geometry B1 (MTH.B406)

Lorentz-Minkowski space

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$(\langle x, y\rangle = 0\ \forall y \Rightarrow x = 0)$

non-degenerate symmetric bilinear form

not positive definite

Euclidean

$\langle\ ,\ \rangle$: The Lorentz inner product on $\mathbb{R}^{n+1}$.

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle := -x_0y_0 + x_1y_1 + \cdots + x_ny_n = \boxed{\boldsymbol{x}^T Y \boldsymbol{y},}$$

+

where $\quad \boldsymbol{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad$ and $\quad Y := \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$

The Lorentz group

$\mathrm{O}(n,1)$: $\ni A$

the group consits of the linear transformations of $\mathbb{R}^{n+1}$ preserving the Lorentz inner product $\langle\ ,\ \rangle$

(n+1)×(n+1) matrices of real components

$$\mathrm{O}(n,1) = \{A \in \mathrm{M}_{n+1}(\mathbb{R}), ; \langle A\boldsymbol{x}, A\boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle \text{ for any } \boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^{n+1}\}$$
$$= \{A \in \mathrm{M}_{n+1}(\mathbb{R})\,;\, \boxed{A^T Y A = Y}\},$$

$$\begin{cases} \langle Ax, Ay\rangle = x^T A^T Y A y \\ \langle x, y\rangle = x^T Y y \end{cases}$$

Lemma

$A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{O}(n,1) \Rightarrow \underline{\det A = \pm 1}$ *and* $|a_{00}| \geqq 1.$

- $A^T Y A = Y$

topleft component of LHS

$$-a_{00}^2 + a_{10}^2 + a_{20}^2 + \dots + a_{n0}^2 = -1$$

$$a_{00}^2 = 1 + a_{10}^2 + \dots + a_{n0}^2 \geq 1.$$

- $\det(A^T Y A) = \det Y \Rightarrow (\det A)^2 = 1$

The Lorentz group

Fact

The set $\mathrm{O}(n,1) \subset \mathrm{M}_{n+1}(\mathbb{R})$ *consits of for* ***connected*** *components,*

$$\boxed{\mathrm{SO}_+(n,1)} := \{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A > 0, a_{00} > 0\};$$

$$\{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A < 0, a_{00} > 0\},$$

$$\{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A > 0, a_{00} < 0\},$$

$$\{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A < 0, a_{00} < 0\}.$$

O(n.1)

$$SO_+(1,1) = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\} \qquad SO(2) = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right\}$$

Lorentz boost (timelike rotation)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO_+(1,1)$$

$$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad ad-bc=1, \ a>0$$

$$\Rightarrow -a^2+c^2=-1, \quad -ab+cd=0; \quad ad-bc=1$$

$$-b^2+d^2=1 \qquad a>0$$

$$\Rightarrow a=\cosh t, \quad c=\sinh t$$

$$b=\sinh t \qquad d=\cosh t,$$

Causality (因果特性) ⇐ word from relativity theory.

Definition

A vector x in $\mathbb{R}^{n+1}$ is said to be

- space-like $\Leftrightarrow \langle x, x\rangle > 0$ or $x = \mathbf{0}$,
- time-like $\Leftrightarrow \langle x, x\rangle < 0$,
- light-like or null or isotoropic $\Leftrightarrow \langle x, x\rangle = 0$ and $x \neq \mathbf{0}$.

$e_0^\perp$

$e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad \boxed{e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \; e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}$

$\langle e_0, e_0\rangle = -1$

timelike

spacelike

$\boxed{\langle\ ,\ \rangle\big|_{e_0^\perp} : \text{positive definite}}$

The orthogonal complement of a time-like vector

Lemma $(\langle x, x\rangle < 0)$

Let $\boldsymbol{x} \in \mathbb{R}^{n+1}$ *be a time-like vector. Then its orthogonal complement*

$$\boldsymbol{x}^{\perp} := \{\boldsymbol{y} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{y}, \boldsymbol{x}\rangle = 0\}$$

is an n*-dimensional linear subspace of* $\mathbb{R}^{n+1}$*, consisting of space-like vectors.* $\langle y, y\rangle > 0$ for $\forall\, y \in x^{\perp} \setminus \{0\}$

$$x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = x_0 e_0 + \vec{x} \qquad y = y_0 e_0 + \vec{y} \in x^{\perp}$$

spacelike

- $\langle x, x \rangle = -x_0^2 + \langle \vec{x}, \vec{x} \rangle < 0 \rightarrow$
- $\langle x, y \rangle = -x_0 y_0 + \langle \vec{x} \cdot \vec{y} \rangle = 0$
- $\langle y, y \rangle = -y_0^2 + \langle \vec{y} \cdot \vec{y} \rangle$

$$= \frac{\langle \vec{x} \cdot \vec{y} \rangle}{x_0} + \langle \vec{y} \cdot \vec{y} \rangle$$

$$= \frac{1}{x_0}\left(\langle \vec{x}, \vec{y} \rangle + x_0 \langle \vec{y} \cdot \vec{y} \rangle \right)$$

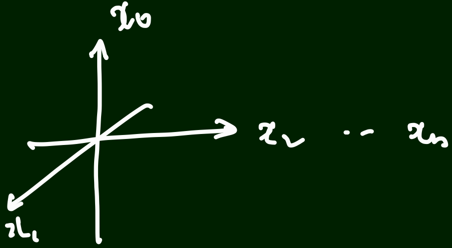
(by Cauchy Schwarz
$|\langle \vec{x}, \vec{y} \rangle|^2 \leq \langle \vec{x}, \vec{x} \rangle \langle \vec{y} \vec{y} \rangle$)

$\boxed{\langle y \cdot y \rangle > 0}$ ...

The Lorentz-Minkowski space

$\mathbb{L}^{n+1}$: the $(n+1)$-dimensional Lorentz-Minkowski space

- $\mathbb{L}^{n+1} = \mathbb{R}^{n+1}$ as a manifold.
- For each $\boldsymbol{x}$, $T_{\boldsymbol{x}}\mathbb{L}^{n+1} = \mathbb{R}^{n+1}$ is endowed with the Lorentz inner product.



Isometries

Fact

An isometry of $\mathbb{L}^{n+1}$ *is in the form*

$$\mathbb{L}^{n+1} \ni \boldsymbol{x} \longmapsto A\boldsymbol{x} + \boldsymbol{a} \in \mathbb{L}^{n+1}, \qquad A \in \mathrm{O}(n \cancel{+1}, 1), \quad \boldsymbol{a} \in \mathbb{R}^{n+1}.$$

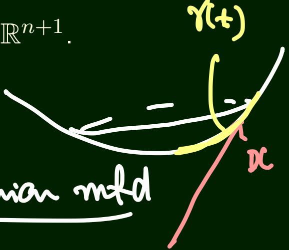
The Hyperbolic space

The hyperbolic space:

$\mathbb{L}^{n+1}$

$$H^n := \{\boldsymbol{x} = (x_0, \ldots, x_n)^T \in \mathbb{R}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\} \subset \mathbb{L}^{n+1}$$

- n-dimensional submanifold of $\mathbb{L}^{n+1} = \mathbb{R}^{n+1}$.
- $T_{\boldsymbol{x}} H^n = \boldsymbol{x}^{\perp}$.
- $\langle \; , \; \rangle$ is positive definite on $T_{\boldsymbol{x}} \mathbb{L}^{n+1}$.

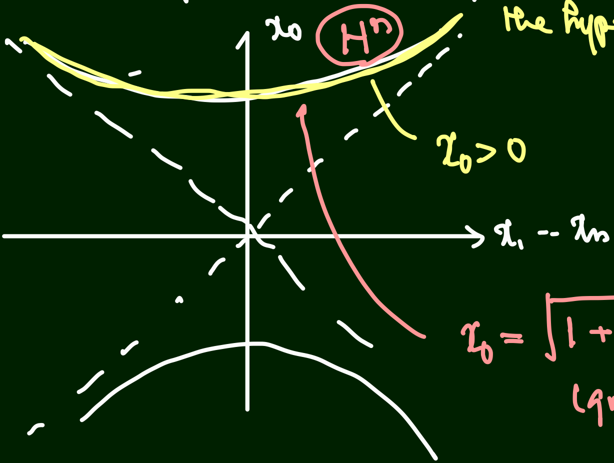


$(H^n, \langle \,,\, \rangle)$: Riemannian mfd

$\langle \gamma(t), \gamma(t) \rangle = -1$

$\Rightarrow \langle \gamma, \gamma' \rangle = 0$, $\langle x, v \rangle = 0$ $\quad \begin{cases} \gamma(0) = x \\ \gamma'(0) = v \in T_x H^n \end{cases}$

$$-x_0^2 + x_1^2 + \cdots + x_n^2 = -1$$



the hyperboloid of 2 sheets.

$x_0 > 0$

$$x_0 = \sqrt{1 + x_1^2 + \cdots x_n^2}$$

(graph)

Completeness

Let $\boldsymbol{x} \in H^n$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} H^n$ with $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$. Then $\langle x, v \rangle = 0$

$$\checkmark \quad \gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}, \quad = \gamma$$

is a geodesic on H^n with $\gamma_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{x}$ and $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{v}$. In particular, the hyperbolic space is complete because the geodesics $\gamma_{\boldsymbol{x},\boldsymbol{v}}$ are defined whole on $\mathbb{R}$.

$$\langle \gamma, \gamma \rangle = \cosh^2 t \underbrace{\langle x, x \rangle}_{-1} + 2\cosh t \sinh t \, \cancel{\langle x, v \rangle} + \sinh^2 t \underbrace{\langle v, v \rangle}_{1} = -1$$

Next week (Final lecture):

- $H^2 \subset \mathbb{L}^3$ is "isometric" of the hyperbolic plane

: models of hyperbolic space

Exercise 6-1

Problem

$A^TYA=Y$ - - -

Let

$\in SO_+(1,1)$

$$A := \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix}, B := \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix},$$

$\in SO(2)$

where u and v are real numbers.

1. *Verify that A and B are elenents of $\mathrm{SO}_+(3,1)$.*
2. *When A is conjugate to B? (Hint: Compute the eigenvalues.)*

Exercise 6-2

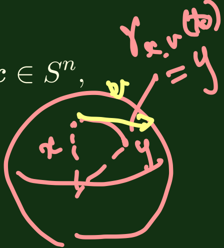
Problem

Euclidean inner product

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x} \cdot \boldsymbol{x} = 1\},$$

$$T_{\boldsymbol{x}}S^n := \textit{the tangent space of } S^n \textit{ at } \boldsymbol{x} \in S^n,$$

$$U_{\boldsymbol{x}}S^n := \{\boldsymbol{v} \in T_{\boldsymbol{x}}S^n\,;\, |\boldsymbol{v}| = 1\}.$$



1. $T_{\boldsymbol{x}}S^n = \boldsymbol{x}^{\perp} = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x} \cdot \boldsymbol{v} = 0\}$.
2. *Show that the curve*

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos t)\boldsymbol{x} + (\sin t)\boldsymbol{v} \qquad (\boldsymbol{x} \in S^n, \boldsymbol{v} \in U_{\boldsymbol{x}}S^n)$$

in $\mathbb{R}^{n+1}$ *is a curve on* S^n *with* $\gamma_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{x}$ *and* $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{v}$,

3. *Let* $\boldsymbol{x}$, $\boldsymbol{y} \in S^n$ $(\boldsymbol{x} \neq \boldsymbol{y})$*: Find* $\boldsymbol{v} \in U_{\boldsymbol{x}}S^n$ *and* $t_0 \in (-\pi, \pi)$ *such that* $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$. *(Hint: orthogonalization)*