# Advanced Topics in Geometry B1 (MTH.B406)

Lorentz-Minkowski space

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2025/07/25 (2023/04/25 訂正)

# The Lorentz inner product

$\langle\ ,\ \rangle$: The Lorentz inner product on $\mathbb{R}^{n+1}$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n = \boldsymbol{x}^T Y \boldsymbol{y},$$

$$\text{where} \quad \boldsymbol{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \text{and} \quad Y := \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

# The Lorentz group

$\mathrm{O}(n,1)$:
the group consits of the linear transformations of $\mathbb{R}^{n+1}$ preserving the Lorentz inner product $\langle\ ,\ \rangle$

$$\begin{aligned}\mathrm{O}(n,1) &= \{A \in \mathrm{M}_{n+1}(\mathbb{R}),;\ \langle A\boldsymbol{x}, A\boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle \text{ for any } \boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^{n+1}\} \\ &= \{A \in \mathrm{M}_{n+1}(\mathbb{R})\,;\ A^T Y A = Y\},\end{aligned}$$

# The Lorentz group

**Lemma**

$A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{O}(n,1) \Rightarrow \det A = \pm 1$ *and* $|a_{00}| \geqq 1$.

# The Lorentz group

**Fact**

*The set* $\mathrm{O}(n,1) \subset \mathrm{M}_{n+1}(\mathbb{R})$ *consits of for connected components,*

$$\begin{aligned}
\mathrm{SO}_+(n,1) :=& \{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A > 0, a_{00} > 0\},\\
&\{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A < 0, a_{00} > 0\},\\
&\{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A > 0, a_{00} < 0\},\\
&\{A = (a_{ij})_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}(\mathbb{R})\,;\det A < 0, a_{00} < 0\}.
\end{aligned}$$

# Causality

## Definition

A vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}$ is said to be

- space-like $\Leftrightarrow \langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ or $\boldsymbol{x} = \boldsymbol{0}$,
- time-like $\Leftrightarrow \langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$,
- light-like or null or isotoropic $\Leftrightarrow \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ and $\boldsymbol{x} \neq \boldsymbol{0}$.

# The orthogonal complement of a time-like vector

**Lemma**

*Let* $\boldsymbol{x} \in \mathbb{R}^{n+1}$ *be a time-like vector. Then its orthogonal complement*

$$\boldsymbol{x}^{\perp} := \{\boldsymbol{y} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{y}, \boldsymbol{x} \rangle = 0\}$$

*is an* $n$*-dimensional linear subspace of* $\mathbb{R}^{n+1}$*, consisting of space-like vectors.*

# The Lorentz-Minkowski space

$\mathbb{L}^{n+1}$: the $(n+1)$-dimensional Lorentz-Minkowski space

- $\mathbb{L}^{n+1} = \mathbb{R}^{n+1}$ as a manifold.
- For each $\boldsymbol{x}$, $T_{\boldsymbol{x}}\mathbb{L}^{n+1} = \mathbb{R}^{n+1}$ is endowed with the Lorentz inner product.

# Isometries

**Fact**

*An isometry of* $\mathbb{L}^{n+1}$ *is in the form*

$$\mathbb{L}^{n+1} \ni \boldsymbol{x} \longmapsto A\boldsymbol{x} + \boldsymbol{a} \in \mathbb{L}^{n+1}, \qquad A \in \mathrm{O}(n+1,1), \quad \boldsymbol{a} \in \mathbb{R}^{n+1}.$$

# The Hyperbolic space

The hyperbolic space:

$$H^n := \{\boldsymbol{x} = (x_0, \ldots, x_n)^T \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\} \subset \mathbb{L}^{n+1}$$

- $n$-dimensional submanifold of $\mathbb{L}^{n+1} = \mathbb{R}^{n+1}$.
- $T_{\boldsymbol{x}} H^n = \boldsymbol{x}^\perp$.
- $\langle\ ,\ \rangle$ is positive definite on $T_{\boldsymbol{x}}\mathbb{L}^{n+1}$.

# Completeness

Let $\boldsymbol{x} \in H^n$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} H^n$ with $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$. Then

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

is a geodesic on $H^n$ with $\gamma_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{x}$ and $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{v}$. In particular, the hyperbolic space is complete because the geodesics $\gamma_{\boldsymbol{x},\boldsymbol{v}}$ are defined whole on $\mathbb{R}$.

# Exercise 6-1

## Problem

*Let*

$$A := \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix}, B := \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix},$$

*where $u$ and $v$ are real numbers.*

1. *Verify that $A$ and $B$ are elenents of $\mathrm{SO}_+(3,1)$.*
2. *When $A$ is conjugate to $B$? (Hint: Compute the eigenvalues.)*

# Exercise 6-2

## Problem

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{x} = 1\},$$
$$T_{\boldsymbol{x}}S^n := \textit{the tangent space of } S^n \textit{ at } \boldsymbol{x} \in S^n,$$
$$U_{\boldsymbol{x}}S^n := \{\boldsymbol{v} \in T_{\boldsymbol{x}}S^n\,;\, |\boldsymbol{v}| = 1\}.$$

1. $T_{\boldsymbol{x}}S^n = \boldsymbol{x}^{\perp} = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{v} = 0\}$.
2. *Show that the curve*

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos t)\boldsymbol{x} + (\sin t)\boldsymbol{v} \qquad \bigl(\boldsymbol{x} \in S^n, \boldsymbol{v} \in U_{\boldsymbol{x}}S^n\bigr)$$

   *in* $\mathbb{R}^{n+1}$ *is a curve on* $S^n$ *with* $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x}$ *and* $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{v}$,
3. *Let* $\boldsymbol{x}$, $\boldsymbol{y} \in S^n$ $(\boldsymbol{x} \neq \boldsymbol{y})$*: Find* $\boldsymbol{v} \in U_{\boldsymbol{x}}S^n$ *and* $t_0 \in (-\pi,\pi)$ *such that* $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$. *(Hint: orthogonalization)*