Advanced Topics in Geometry B1 (MTH.B406)

Hyperbolic Space

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2025/08/01

Exercise 6-1

Problem

$A \in O(3,1)$

$B \in$

Let a_0

$$A := \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix}, B := \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix},$$

where u and v are real numbers.

1. *Verify that A and B are elenents of $\mathrm{SO}_+(3,1)$.*
2. *When A is conjugate to B? (Hint: Compute the eigenvalues.)*

$\langle a_0, a_0 \rangle = -\frac{9}{4} + 1 + \frac{1}{4} = -1$ ⋯ $-$

det $\quad a_{00}$

Exercise 6-1

$SO_+(3,1) = \{A \in O(3,1);\ \det A > 0,\ a_{00} > 0\}$

column vect's

Fact

A 4×4*-matrix* $A = (\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ *is an element of* $\mathrm{O}(3,1)$ *if and only if*

$$\langle \boldsymbol{a}_i, \boldsymbol{a}_i \rangle = \begin{cases} -1 & (i = 0) \\ 1 & (i = \cancel{2,3,4}\ 1\ 2\ 3) \end{cases}.$$

and $\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle = 0$ *when* $i \neq j$.

$$A^T Y A = Y \qquad Y = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\{a_0\ a_1\ a_2\ a_4\}$: a Lorentzian orthonormal system.

Exercise 6-1

almost all matrix is conj to B. normal form

$$A = \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix} \qquad B := \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix}$$

Eigenvalues: 1 mult. 4 Eigenvalues: $e^{\pm u}$, $e^{\pm iv}$

If A is conj. to B $\Rightarrow$ $\forall$ eigenvalues of B are 1

$\Rightarrow$ $u = v = 0 \Rightarrow B = id$

A is conjugate to B $\Leftrightarrow$ $\exists P$ s.t $P^{-1}AP = B$

Is $B = id$ conjugate to A?

No If so $P^{-1}AP = id$

$A = PP^{-1} = id$

contradiction.

Answer Never!

Exercise 6-1

Fact

A matrix $A \in \mathrm{SO}(4)$ is conjugate to a matrix

$$\begin{pmatrix} \cos u & -\sin u & 0 & 0 \\ \sin u & \cos u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix}.$$

$SO_e(3,1)$

$$\begin{pmatrix} e^u & & & \\ & e^{-u} & & \\ & & e^{v} & \\ & & & e^{-v} \end{pmatrix}$$

$$\begin{pmatrix} e^{iu} & 0 & & \\ 0 & e^{-iu} & & \\ & & e^{iv} & 0 \\ & & 0 & e^{-iv} \end{pmatrix}$$

Eigenvalues: e^{iu} e^{-iu} e^{iv} e^{-iv}

A : a normal matrix $\qquad A\bar{A}^T = \bar{A}^T A$

$\Rightarrow$ diagonalizable $\qquad \left(AA^T = A^T A = \mathrm{id} \right)$

Exercise 6-2

Problem

$\subset \mathbb{R}^{n+1}$

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{x} = 1\},$$ unit sphere

- *$T_{\boldsymbol{x}}S^n :=$ the tangent space of S^n at $\boldsymbol{x} \in S^n$,*
- $U_{\boldsymbol{x}}S^n := \{\boldsymbol{v} \in T_{\boldsymbol{x}}S^n\,;\, \underline{|\boldsymbol{v}| = 1}\}$.

1. $T_{\boldsymbol{x}}S^n = \boldsymbol{x}^\perp = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{v} = 0\}$.
2. *Show that the curve*

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos t)\boldsymbol{x} + (\sin t)\boldsymbol{v} \qquad \big(\boldsymbol{x} \in S^n, \boldsymbol{v} \in U_{\boldsymbol{x}}S^n\big)$$

in $\mathbb{R}^{n+1}$ is a curve on S^n with $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x}$ and $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{v}$,

3. *Let $\boldsymbol{x}$, $\boldsymbol{y} \in S^n$ $(\boldsymbol{x} \neq \pm\boldsymbol{y})$: Find $\boldsymbol{v} \in U_{\boldsymbol{x}}S^n$ and $t_0 \in (-\pi, \pi)$ such that $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$. (Hint: orthogonalization)*

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{x} = 1\},$$
$$U_{\boldsymbol{x}}S^n := \{\boldsymbol{v} \in T_{\boldsymbol{x}}S^n\,;\, |\boldsymbol{v}| = 1\}.$$

1. $T_{\boldsymbol{x}}S^n = \boldsymbol{x}^\perp = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{v} = 0\}.$

$\gamma(t)$: a curve in S^n ; $\gamma(0) = x$, $\gamma'(0) = v$

$$\overset{0}{\cancel{1}} = \left(\gamma(\overset{0}{\cancel{t}})\cdot\gamma(\overset{0}{\cancel{t}})\right)' = 2\gamma(\overset{0}{\cancel{t}})\cdot\gamma'(\overset{0}{\cancel{t}}) = 2x\cdot v$$

$v \in x^\perp$

$$\underset{n\text{-dim}}{T_xS^n} \underset{=}{\cancel{\subset}} \underset{n\text{-dim}}{x^\perp}$$

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x} \cdot \boldsymbol{x} = 1\},$$
$$T_{\boldsymbol{x}} S^n := \text{the tangent space of } S^n \text{ at } \boldsymbol{x} \in S^n.$$

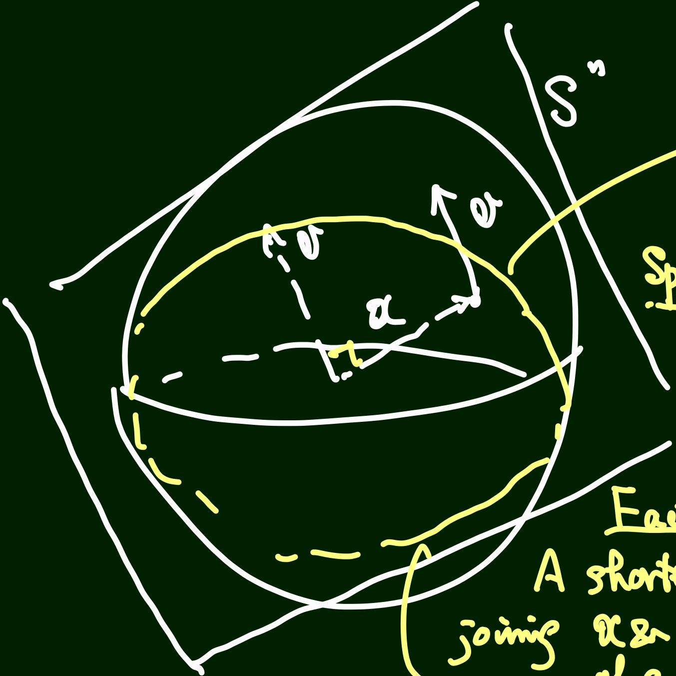
2. Show that the curve

$|v| = 1$

$$\gamma(t) = \gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos t)\boldsymbol{x} + (\sin t)\boldsymbol{v} \qquad (\boldsymbol{x} \in S^n, \boldsymbol{v} \in U_{\boldsymbol{x}} S^n)$$

in $\mathbb{R}^{n+1}$ is a curve on S^n with $\gamma_{\boldsymbol{x},\boldsymbol{v}}(\overset{0}{\cancel{t}}) = \boldsymbol{x}$ and $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(\overset{0}{\cancel{t}}) = \boldsymbol{v}$,

- $\gamma(0) = x \quad \gamma'(0) = v$
- $\gamma \cdot \gamma = 1 \quad (\because |x| = 1,\ x \cdot v = 0,\ |v| = 1)$



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great circle

$\mathrm{Span}\{x, v\} \cap S^n$
$\parallel$
$\gamma_{x,v}(\mathbb{R})$

<u>Fact</u>

A shortest path joining x & y: an arc of a great circle

Exercise 6-2

3. Let $\boldsymbol{x}$, $\boldsymbol{y} \in S^n$ ($\boldsymbol{x} \neq \pm\boldsymbol{y}$): Find $\boldsymbol{v} \in U_{\boldsymbol{x}}S^n$ and $t_0 \in (-\pi, \pi)$ such that $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$.

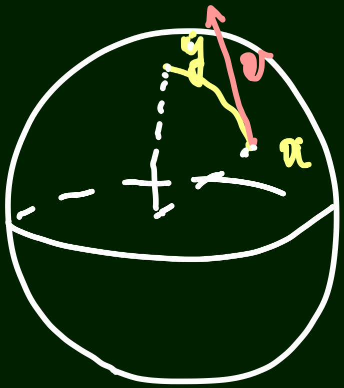
$v \in x^{\perp} \quad |v| = 1$

$y \in \mathrm{Span}\{x, v\}$

$v \in \mathrm{Span}\{x, y\}$

$$\Rightarrow \quad v = \frac{y - (x \cdot y)x}{|y - (x \cdot y)x|}$$

Gram-Schmidt



$$\gamma(t) = (\cos t)\,x + (\sin t)\,v$$

$$\gamma(t_0) = (\cos t_0)\,x + (\sin t_0)\,v = y$$

$$(\cos t_0)\underbrace{(x \cdot x)}_{1} + (\sin t_0)\,\underbrace{\cancel{v \cdot x}}_{0} = x \cdot y$$

arc length ($\because |\gamma'| = 1$)

$$\cos t_0 = x \cdot y$$

$$\mathrm{dist}(x, y) = t_0 = \cos^{-1}(x \cdot y)$$

Distance of Tokyo and Paris

θ

$R(\cos u \cos v, \cos u \sin v, \sin u)$

O-okayama Campus

N35.60447°, E139.68386°

u $\quad$ v

Paris (Tour Eiffel)

y

$R(\sim\sim \quad - \quad)$

N48.85848°, E2.29448°

$R \times \cos^{-1}(x \cdot y)$

Nominal earth radius:

$$R = 6.3781 \times 10^6\text{m}$$

Distance between O-okayama and Tour Eiffel

$$9.7332 \times 10^6\text{m}.$$