Advanced Topics in Geometry B1 (MTH.B406)

Hyperbolic Space

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Exercise 6-1

Problem

Let

$$A := \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix}, B := \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix},$$

where u and v are real numbers.

1. *Verify that A and B are elenents of $\mathrm{SO}_+(3,1)$.*
2. *When A is conjugate to B? (Hint: Compute the eigenvalues.)*

Exercise 6-1

Fact

A 4×4*-matrix* $A = (\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ *is an element of* $\mathrm{O}(3,1)$ *if and only if*

$$\langle \boldsymbol{a}_i, \boldsymbol{a}_i \rangle = \begin{cases} -1 & \qquad\qquad\qquad (i = 0) \\ 1 & (i = 2, 3, 4) \end{cases}$$

and $\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle = 0$ *when* $i \neq j$.

Exercise 6-1

$$A = \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix} \quad B := \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix}$$

Eigenvalues: 1

Eigenvalues: $e^{\pm u}$, $e^{\pm iv}$

Exercise 6-1

Fact

A matrix $A \in \mathrm{SO}(4)$ *is conjugate to a matrix*

$$\begin{pmatrix} \cos u & -\sin u & 0 & 0 \\ \sin u & \cos u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix}.$$

Exercise 6-2

Problem

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{x} = 1\},$$
$$T_{\boldsymbol{x}}S^n := \textit{the tangent space of } S^n \textit{ at } \boldsymbol{x} \in S^n,$$
$$U_{\boldsymbol{x}}S^n := \{\boldsymbol{v} \in T_{\boldsymbol{x}}S^n\,;\, |\boldsymbol{v}| = 1\}.$$

1. $T_{\boldsymbol{x}}S^n = \boldsymbol{x}^{\perp} = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{v} = 0\}$.
2. *Show that the curve*

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos t)\boldsymbol{x} + (\sin t)\boldsymbol{v} \qquad \big(\boldsymbol{x} \in S^n, \boldsymbol{v} \in U_{\boldsymbol{x}}S^n\big)$$

   in $\mathbb{R}^{n+1}$ *is a curve on* S^n *with* $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x}$ *and* $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{v}$,
3. *Let* $\boldsymbol{x}$, $\boldsymbol{y} \in S^n$ $(\boldsymbol{x} \neq \pm\boldsymbol{y})$*: Find* $\boldsymbol{v} \in U_{\boldsymbol{x}}S^n$ *and* $t_0 \in (-\pi, \pi)$ *such that* $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$*. (Hint: orthogonalization)*

Exercise 6-2

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x} \cdot \boldsymbol{x} = 1\},$$
$$U_{\boldsymbol{x}} S^n := \{\boldsymbol{v} \in T_{\boldsymbol{x}} S^n\,;\, |\boldsymbol{v}| = 1\}.$$

1. $T_{\boldsymbol{x}} S^n = \boldsymbol{x}^\perp = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x} \cdot \boldsymbol{v} = 0\}$.

Exercise 6-2

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{x} = 1\},$$
$$T_{\boldsymbol{x}}S^n := \text{the tangent space of } S^n \text{ at } \boldsymbol{x} \in S^n.$$

2. Show that the curve

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos t)\boldsymbol{x} + (\sin t)\boldsymbol{v} \qquad \big(\boldsymbol{x} \in S^n, \boldsymbol{v} \in U_{\boldsymbol{x}}S^n\big)$$

in $\mathbb{R}^{n+1}$ is a curve on S^n with $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x}$ and $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{v}$,

Exercise 6-2

3. Let $\boldsymbol{x}$, $\boldsymbol{y} \in S^n$ $(\boldsymbol{x} \neq \pm\boldsymbol{y})$: Find $\boldsymbol{v} \in U_{\boldsymbol{x}}S^n$ and $t_0 \in (-\pi, \pi)$ such that $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$.