Advanced Topics in Geometry B1 (MTH.B406)

Hyperbolic Space

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Hyperbolic plane

work on general dimensions

$(x_0-x_1)(x_0+x_1) = x_0^2 - x_1^2 = 1 + x_2^2 > 0$

$x_0 + x_1 > 0$

does not change sign

$$H^2 := \{\boldsymbol{x} = (x_0, x_1, x_2)^T \in \mathbb{L}^3 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\} \ni (1,0,0)$$

$$\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 \,;\, y > 0\}, \qquad ds^2 = \frac{dx^2 + dy^2}{y^2},$$

$y > 0$

$$\pi \colon H^2 \ni (x_0, x_1, x_2) \longmapsto \left(\overset{x}{\frac{x_2}{x_0 + x_1}}, \overset{y}{\frac{1}{x_0 + x_1}} \right) \in \mathbb{R}^2_+.$$

(diffeo)

- π is an isometry.

∃ inverse

$x_2 = \dfrac{x}{y}$

$x_0 + x_1 = \dfrac{1}{y}$

$x_0 - x_1 = \dfrac{1 + x_2^2}{x_0 + x_1} = \dfrac{x^2 + y^2}{y}$

$$x_0 = \frac{1+x^2+y^2}{2y}. \quad x_1 = \frac{1-x^2-y^2}{2y}, \quad x_2 = \frac{x}{y}$$

$$\underbrace{-dx_0^2 + dx_1^2 + dx_2^2}_{\text{Lorentzian}}$$

$$= \frac{dx^2+dy^2}{y^2} = \underline{ds^2}$$

$$H^2 \cong (\mathbb{R}^2_+, ds^2)$$

as Riem manifolds

Straight lines

Lemma

Let

$$\sigma(t) = \sigma_{c,r}(t) := \begin{cases} (r\tanh t + c, r\,\mathrm{sech}\, t) & (0 < r < \infty), \\ (c, e^t) & (r = +\infty), \end{cases}$$

where $c \in \mathbb{R}$. *Then*

$$\pi^{-1} \circ \sigma(t) = (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

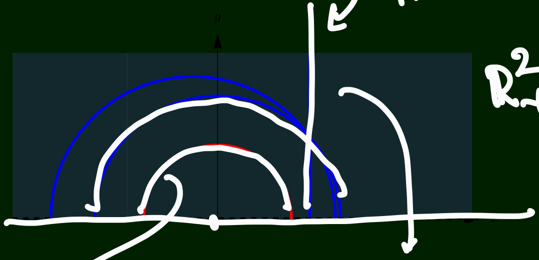
"great circle" on H^2

in $H^2 \subset L^3$

for some $\boldsymbol{x} \in H^2$ *and* $\boldsymbol{v} \in T_{\boldsymbol{x}} H^2$ *with* $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$.

Straight lines

upper half plane.



$\mathbb{R}^2_+$

- $\sigma(t) = (c, e^t)$
- $\sigma(t) = (\underline{r} \underline{\tanh} t + \underline{c}, \underline{r} \underline{\operatorname{sech} t})$

Straight lines

- $T_{\boldsymbol{x}} H^2 = \boldsymbol{x}^{\perp}$
- $U_{\boldsymbol{x}} H^2 = \{\boldsymbol{v} \in T_{\boldsymbol{x}} H^2\,;\, \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1\}$.

$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \underline{(\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}.}$ ← "shortest path"

- γ is a curve in H^2 with $\gamma(0) = \boldsymbol{x}$, $\gamma'(0) = \boldsymbol{v}$.
- $\langle \gamma', \gamma' \rangle = 1$, i.e., t is the arc-length parameter.

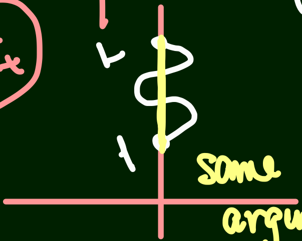
Straight line

Proposition

~~A~~ The shortest path joining two distinct points x and $y \in H^2$ is parametrized as $\gamma_{x,v}$, where

$$v = \frac{y + \langle x, y \rangle x}{|y + \langle x, y \rangle x|} \in U_x H^2$$

in $\mathbb{R}^2_+$



Gram-Schmidt orth. of $\{x, y\}$.

Note: $\langle x, x \rangle = -1$

same argument of Prop 5-2

Distance

Proposition

For $\boldsymbol{x}$, $\boldsymbol{y} \in H^2$,

- $$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^{-1}(-\underbrace{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}_{\boxed{\leq -1}}).$$

Pythagorean theorem

$$\underline{\cosh X = \cosh Y \cosh Z}$$

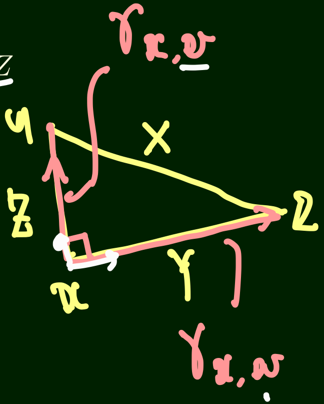
$$v = \frac{y + \langle x, y\rangle x}{|\cdot|}$$

$$w = \frac{z + \langle x, z\rangle x}{|\cdot|}$$

$v \perp w$

$0 = \langle v, w\rangle \Rightarrow \underline{-\langle y, z\rangle}$

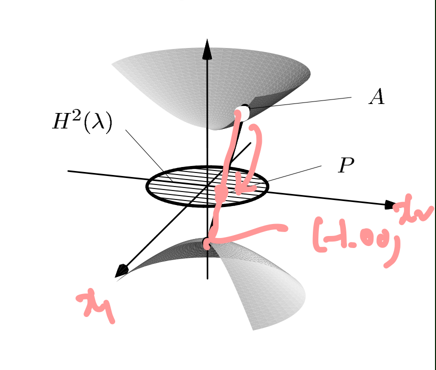
$= (-\langle x, y\rangle)(-\langle x, z\rangle)$



Poincaré disc model

Stereographic projection: (cf. $S^2 \setminus \{1\text{ point}\} \to \mathbb{C}$)

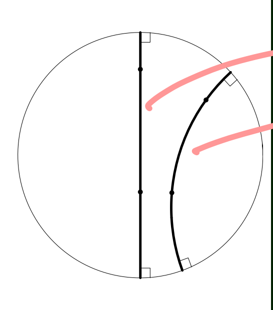
- $D := \{(u, v)\,;\, u^2 + v^2 < 1\}$.
- $\pi_P \colon H^2 \ni (x_0, x_1, x_2) \mapsto (u, v) = \frac{1}{1+x_0}(x_1, x_2) \in D$.



Poincaré disc model

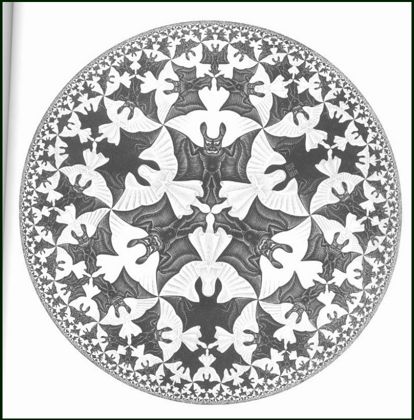
Poincaré disc model:

$$D = \{(u,v)\,;\, u^2+v^2<1\}, \qquad ds^2 = \frac{4}{(1-u^2-v^2)^2}(du^2+dv^2).$$



lines

Poincaré disc model



Circle limit IV (M.C. Escher, 1960)

Klein model

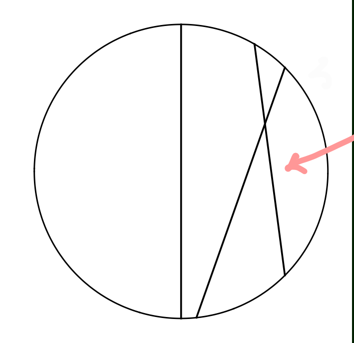
"projective model"

Central projection:

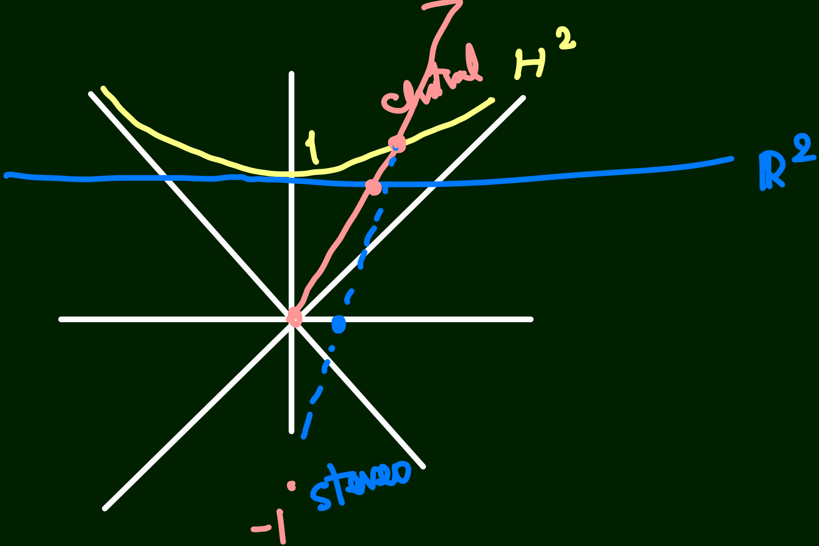
19c.

- $D := \{(\xi, \eta)\,;\, \xi^2 + \eta^2 < 1\}$.
- $\pi_C\colon H^2 \ni (x_0, x_1, x_2) \mapsto (\xi, \eta) = \frac{1}{x_0}(x_1, x_2) \in D$. Beltrami

$$ds_C^2 = \frac{1}{(1 - \xi^2 - \eta^2)}\big((1-\eta^2)\,d\xi^2 + 2\xi\eta\,d\xi\,d\eta + (1-\xi^2)\,d\eta^2\big).$$



straight lines



central
H^2
1
$\mathbb{R}^2$
stereo
-1

Final Remarks

- Non-Euclidean geometry

  Virtual object. ,,,, (upper half plane model)

- Pseudospherical surfaces

  Locally realizable E. Beltrami: not complete

- Construction of Pseudospherical surfaces

  Integrable systems. ∃ many local models

- Non-existence of models of non-Euclidean geometry as surfaces in $\mathbb{R}^3$ (Hilbert)

  ∄ global models.

- A model of non-Euclidean geometry as a surface in Lorentz-Minkowski space.

  hyperboloid

- Various models of non-Euclidean geometry.

Enjoy!