Advanced Topics in Geometry B1 (MTH.B406)

Hyperbolic Space

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Hyperbolic plane

$$H^2 := \{\boldsymbol{x} = (x_0, x_1, x_2)^T \in \mathbb{L}^3\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}$$

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2\,;\, y > 0\}, \qquad ds^2 = \frac{dx^2 + dy^2}{y^2},$$

$$\pi\colon H^2 \ni (x_0, x_1, x_2) \longmapsto \left(\frac{x_2}{x_0 + x_1}, \frac{1}{x_0 + x_1}\right) \in \mathbb{R}^2_+.$$

- π is an isometry.

Straight lines

Lemma

Let

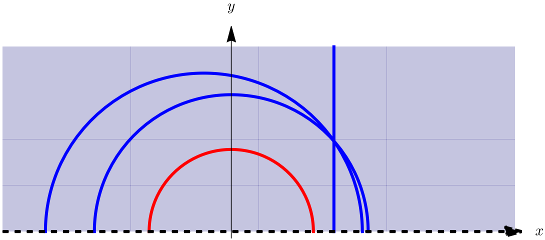
$$\sigma(t) = \sigma_{c,r}(t) := \begin{cases} (r\tanh t + c, r\,\mathrm{sech}\,t) & (0 < r < \infty), \\ (c, e^t) & (r = +\infty), \end{cases}$$

where $c \in \mathbb{R}$. *Then*

$$\pi^{-1} \circ \sigma(t) = (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

for some $\boldsymbol{x} \in H^2$ *and* $\boldsymbol{v} \in T_{\boldsymbol{x}}H^2$ *with* $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$.

Straight lines



Straight lines

- $T_{\boldsymbol{x}} H^2 = \boldsymbol{x}^{\perp}$
- $U_{\boldsymbol{x}} H^2 = \{\boldsymbol{v} \in T_{\boldsymbol{x}} H^2\,;\, \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1\}$.

$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$.

- γ is a curve in H^2 with $\gamma(0) = \boldsymbol{x}$, $\gamma'(0) = \boldsymbol{v}$.
- $\langle \gamma', \gamma' \rangle = 1$, i.e., t is the arc-length parameter.

Straight line

Proposition

A shortest path joining two distinct points $\boldsymbol{x}$ and $\boldsymbol{y} \in H^2$ is parametrized as $\gamma_{\boldsymbol{x},\boldsymbol{v}}$, where

$$\boldsymbol{v} = \frac{\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{x}}{|\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{x}|} \in U_{\boldsymbol{x}} H^2$$

Distance

Proposition

For $\boldsymbol{x}$, $\boldsymbol{y} \in H^2$,

$$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \cosh(-\langle \boldsymbol{x}, \boldsymbol{y} \rangle).$$

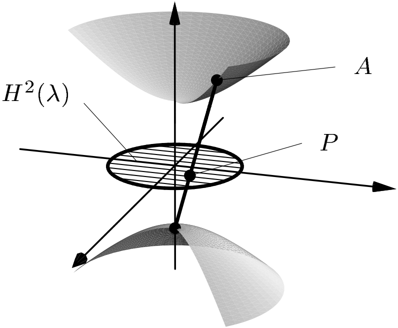
Pythagorean theorem

$$\cosh X = \cosh Y \cosh Z$$

Poincaré disc model

Stereographic projection:

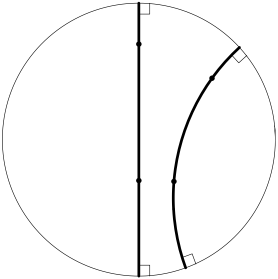
- $D := \{(u, v)\,;\, u^2 + v^2 < 1\}$.
- $\pi_P \colon H^2 \ni (x_0, x_1, x_2) \mapsto (u, v) = \frac{1}{1+x_0}(x_1, x_2) \in D$.



Poincaré disc model

Poincaré disc model:

$$D = \{(u,v)\,;\, u^2+v^2=1\}, \qquad ds^2 = \frac{4}{(1+u^2+v^2)^2}(du^2+dv^2).$$



Klein model

Central projection:

- $D := \{(\xi, \eta)\,;\, \xi^2 + \eta^2 < 1\}$.
- $\pi_C\colon H^2 \ni (x_0, x_1, x_2) \mapsto (\xi, \eta) = \frac{1}{x_0}(x_1, x_2) \in D$.

$$ds_C^2 = \frac{1}{(1-\xi^2-\eta^2)}\big((1-\eta^2)\,d\xi^2 + 2\xi\eta\,d\xi\,d\eta + (1-\xi^2)\,d\eta^2\big).$$