3 Pseudospherical surfaces and asymptotic Chebyshev net

Preliminaries

Let U and V be domains of $\mathbb{R}^n$

Definition 3.1. A C^∞ bijection $\varphi\colon V \to U$ is said to be a *diffeomorphism* if its inverse is also of class C^∞.

Lemma 3.2. *If $\varphi\colon V \to U$ is a diffeomorphism,*

$$(D\varphi)_{\varphi^{-1}(q)} \circ \bigl(D(\varphi^{-1})\bigr)_q = \mathrm{id}_{\mathbb{R}^n}, \qquad \textit{and} \qquad \bigl(D(\varphi^{-1})\bigr)_{\varphi(p)} \circ (D\varphi)_p = \mathrm{id}_{\mathbb{R}^n}$$

hold at each point of $q \in U$ and $p \in V$, where $D\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ and $D(\varphi^{-1})\colon \mathbb{R}^m \to \mathbb{R}^n$ denote the differentials of the map φ and φ^{-1}. $(D\varphi)_p$ is a non-singular matrix on each point of $p \in V$.

Remark 3.3. Define $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) = \varphi(\xi, \eta) = (\xi^3, \eta)$. Then the Jacobi matrix $D\varphi$ is computed as

$$D\varphi = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \begin{pmatrix} 2\xi^2 & 0 \\ 0 & 1 \end{pmatrix}$$

which is singular at the origin. Hence φ is not a diffeomorphism though it is a bijection.

Theorem 3.4 (The inverse function theorem)**.** *Let $\varphi\colon U \to \mathbb{R}^n$ be a C^∞-map defined on a domain $U \subset \mathbb{R}^n$ and $p \in U$. Assume $(D\varphi)_p$ is non-singular. Then there exists a neighborhood $V \subset U$ of p such that $\varphi|_V\colon V \to \varphi(V)$ is a diffeomorphism. Moreover, $\bigl(D(\varphi^{-1})\bigr)_{\varphi(q)} = (D\varphi)_q^{-1}$ holds for each $q \in V$.*

Change of Parameters

Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface in $\mathbb{R}^3$ and $\varphi\colon V \to U$ a diffeomorphism, where U and V are domains of $\mathbb{R}^2$. Then

$$\tilde{p} := p \circ \varphi\colon V \to \mathbb{R}^3 \tag{3.1}$$

gives another regular parametrization of a surface, whose image coincides with that of p. Such $\tilde{p}$ is said to be a parametrized surface obtained by the *coordinate change* φ of p.

Now we write $\varphi\colon (\xi, \eta) \to (u, v)$. Then by the chain rule, it holds that

$$(\tilde{p}_\xi, \tilde{p}_\eta) = (u_\xi p_u + v_\xi p_v, u_\eta p_u + v_\eta p_v) = (p_u, p_v)J, \qquad \text{where} \quad J := D\varphi = \begin{pmatrix} u_\xi & u_\eta \\ v_\xi & v_\eta \end{pmatrix}, \tag{3.2}$$

here p_u, p_v, $\tilde{p}_\xi$, $\tilde{p}_\eta$ are considered to be functions valued in the column-vectors.

We write the first fundamental form ds^2 (resp. $d\tilde{s}^2$) and the second fundamental form $I\!I$ (resp. $\widetilde{I\!I}$) of p (resp. $\tilde{p}$) as

$$\begin{aligned} ds^2 &= E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad & I\!I &= L\,du^2 + 2M\,du\,dv + N\,dv^2 \\ d\tilde{s}^2 &= \widetilde{E}\,d\xi^2 + 2\widetilde{F}\,d\xi\,d\eta + \widetilde{G}\,d\eta^2, \qquad & \widetilde{I\!I} &= \widetilde{L}\,d\xi^2 + 2\widetilde{M}\,d\xi\,d\eta + \widetilde{N}\,d\eta^2 \end{aligned}$$

Since the unit normal vector $\tilde{\nu}$ of $\tilde{p}$ coincides with $\nu \circ \varphi$, (3.2) yield

$$\begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} = J^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J, \qquad \begin{pmatrix} \widetilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix} = J^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} J.$$

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This means that one obtain

$$ds^2 = d\tilde{s}^2, \qquad I\!I = \widetilde{I\!I}$$

by substituting

$$du = u_\xi\, d\xi + u_\eta\, d\eta, \qquad dv = v_\xi\, d\xi + v_\eta\, d\eta.$$

In other word, *the first and second fundamental forms are invariant under changes of parameters.* Moreover, the Gaussian curvature $K = (LN - M^2)/(EG - F^2)$ is also invariant under change of parameters.

Asymptotic parameters

For a surface of negative Gaussian curvature, there exists a parameter such that its second fundamental matrix is anti-diagonal, called an *asymptotic coordinate system.* In other words, a parameter (u, v) is an asymptotic coordinate system if and only if the second fundamental form is in the form

$$I\!I = 2M\, du\, dv.$$

To prove this fact, we prepare

Lemma 3.5. *Let $\omega = \alpha\, du + \beta\, dv$ be a 1-form defined on a domain U of the uv-plane $\mathbb{R}^2$, where α and β are functions in (u, v). Assume $(\alpha, \beta) \neq (0, 0)$ at $\mathrm{P} \in U$. Then there exists a neighborhood $V \subset U$ of P and functions φ and ξ on V such that*

$$\varphi\omega = d\xi, \qquad \varphi(\mathrm{Q}) \neq 0 \quad \textit{for} \quad \mathrm{Q} \in V.$$

Proof. Let $\gamma(s) = \big(u_0(s), v_0(s)\big)$ a curve on U defined on an interval $I := (-\varepsilon, \varepsilon)$ $(\varepsilon > 0)$ satisfying $\gamma(0) = \mathrm{P}$, $\gamma'(s) \neq 0$ $(s \in I)$, and $\gamma'(0) = \big(u_0'(0), \dot{v}_0'(0)\big)$ satisfies

$$\alpha(\mathrm{P})u_0'(0) + \beta(\mathrm{P})v_0'(0) \neq 0. \tag{3.3}$$

Then for each $s \in I$, there exists a solution $\big((u^s(t), v^s(t)\big)$ $(t \in (-\delta_s, \delta_s))$ of a system of ordinary differential equations

$$\frac{d}{dt}u_s(t) = -\beta(u_s(t), v_s(t)), \quad \frac{d}{dt}v_s(t) = \alpha(u_s(t), v_s(t)), \quad u_s(0) = u(s), \quad v_s(0) = v(s).$$

Then, by a regularity of the solution of ordinary differential equations with respect to parameters, we obtain a smooth map

$$(s, t) \mapsto \big(u(s, t), v(s, t)\big) := \big(u_s(t), v_s(t)\big).$$

In particular,

$$\big(u(0,0), v(0,0)\big) = \mathrm{P}, \ \frac{\partial u}{\partial s}(0,0) = u_0'(0), \ \frac{\partial v}{\partial s}(0,0) = v_0'(0), \ \frac{\partial u}{\partial t}(0,0) = -\beta(\mathrm{P}), \ \frac{\partial v}{\partial t}(0,0) = \alpha(\mathrm{P})$$

hold. Hence by (3.3),

$$\det\begin{pmatrix} \frac{\partial u}{\partial s}(0,0) & \frac{\partial u}{\partial t}(0,0) \\ \frac{\partial v}{\partial s}(0,0) & \frac{\partial v}{\partial t}(0,0) \end{pmatrix} = \det\begin{pmatrix} u_0'(0) & -\beta(\mathrm{P}) \\ v_0'(0) & \alpha(\mathrm{P}) \end{pmatrix} \neq 0.$$

Thus, by the inverse function theorem, there exists a neighborhood V of P such that the map $(s, t) \mapsto (u, v)$ is a diffeomorphism, that is, (s, t) is a new coordinate system on $V \subset \mathbb{R}^2$. Using this parameter, we can write

$$\begin{aligned} \omega = \alpha\, du + \beta\, dv &= \alpha\left(\frac{\partial u}{\partial s}\, ds + \frac{\partial u}{\partial t}\, dt\right) + \beta\left(\frac{\partial v}{\partial s}\, ds + \frac{\partial v}{\partial t}\, dt\right) \\ &= \alpha(-\beta\, ds + u_t\, dt) + \beta(\alpha\, ds + v_t\, dt) = (u_t\alpha + v_t\beta)dt. \end{aligned}$$

So, by setting $\varphi := 1/(u_t\alpha + v_t\beta)$ and $\xi = t$, we have the conclusion. □

Remark 3.6. Lemma 3.5 implies that any 1-form on a domain of $\mathbb{R}^2$ is locally a non-zero function multiple of an exact 1-form. The function φ in is called an *integrating factor* of the form ω.

Remark 3.7. Lemma 3.5 is the special (2-dimensional) case of *Caratheodory's principle*, which is often referred in the context of thermodynamics. In fact, Caratheodory's principle says that for any 1-form ω on n-manifold (or $\mathbb{R}^n$), there exists an integrating factor if and only if $\omega \wedge d\omega \neq 0$.

Proposition 3.8 (Asymptotic Coordinate system)**.** *Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface in $\mathbb{R}^3$ whose Gaussian curvature is negative on U. Then for each $\mathrm{P} \in U$, there exists an asymptotic coordinate system on a neighborhood of* P.

Proof. Write the second fundamental form of p as $I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2$. Since the Gaussian curvature is negative, $-\kappa^2 := LN - M^2$ is negative. When $L(\mathrm{P}) = 0$, setting $u = \frac{1}{\sqrt{2}}(s-t)$, $v = \frac{1}{\sqrt{2}}(s+t)$ we get

$$I\!I(\mathrm{P}) = 2M(\mathrm{P})\,du\,dv = M(\mathrm{P})(ds - dt)(ds + dt) = M\,ds^2 - M\,dt^2.$$

Since $L(\mathrm{P}) = 0$, $\kappa(\mathrm{P})^2 = M^2(\mathrm{P}) \neq 0$, and hence the first coefficient of $I\!I$ with respect to the coordinate system (s,t) is not zero. Thus, we may assume $L \neq 0$ holds on a neighborhood of P, without loss of generality.

When $L \neq 0$,

$$\begin{aligned} I\!I &= L\left(du + \frac{M}{L}\,dv\right)^2 + \frac{LN - M^2}{L}dv^2 = L\left(\left(du + \frac{M}{L}\,dv\right)^2 - \left(\frac{\kappa}{L}\,dv\right)^2\right) \\ &= L\left(du + \frac{M+\kappa}{L}\,dv\right)\left(du + \frac{M-\kappa}{L}\,dv\right) \end{aligned}$$

Then by Lemma 3.5, there exists functions ξ, η, φ and ψ such that $\varphi(\mathrm{P}) \neq 0$, $\psi(\mathrm{P}) \neq 0$ and

$$du + \frac{M+\kappa}{L}\,dv = \varphi\,d\xi, \quad du + \frac{M-\kappa}{L}\,dv = \psi\,d\eta.$$

Here

$$\det\begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix} = \frac{1}{\varphi\psi}\det\begin{pmatrix} 1 & \frac{M+\kappa}{L} \\ 1 & \frac{M-\kappa}{L} \end{pmatrix} = \frac{1}{\varphi\psi}\frac{2\kappa}{L} \neq 0$$

holds at P. Hence $(s,t) \mapsto (\xi,\eta)$ is a change of coordinates, and

$$I\!I = 2\widetilde{M}\,d\xi\,d\eta, \qquad (2\widetilde{M} = L\varphi\psi).$$

So (ξ,η) is an asymptotic coordinate system. □

Asymptotic Chebyshev net

Theorem 3.9. *For a each point* P *of a surface of constant negative Gaussian curvature $-k^2$, there exists a neighborhood U of* P *and coordinate system (ξ,η) such that the first and second fundamental forms are in the form*

$$ds^2 = d\xi^2 + 2\cos\theta\,d\xi\,d\eta + d\eta^2, \qquad I\!I = 2k\sin\theta\,d\xi\,d\eta, \tag{3.4}$$

where θ is a smooth function in (ξ,η) with $0 < \theta(\xi,\eta) < \pi$.

Proof. By Proposition 3.8, there exists an asymptotic coordinate system (u,v) around P:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad I\!I = 2M\,du\,dv.$$

Then by the result in Exercise 5-1 of MTH.B405[2], $E_v = G_u = 0$ holds. Since both $E = p_u \cdot p_u$ and $G = p_v \cdot p_v$ are positive, we can write

$$E\,du^2 = \bigl(e(u)\,du\bigr)^2, \qquad G\,dv^2 = \bigl(g(v)\,dv\bigr)^2,$$

where $e(u)$ and $g(v)$ are positive functions in u and v, respectively. Set

$$\xi = \xi(u) = \int_{u_0}^{u} e(t)\,dt, \qquad \eta = \eta(v) = \int_{v_0}^{v} g(t)\,dt,$$

where $\mathrm{P} = (u_0, v_0)$. Then the map $(u, v) \mapsto (\xi(u), \eta(v))$ is a coordinate change because e and g are positive, and the first fundamental form and second fundamental form are written as

$$ds^2 = d\xi^2 + 2\widetilde{F}\,d\xi\,d\eta + d\,\eta^2, \quad I\!I = 2\widetilde{M}\,d\xi\,d\eta.$$

Since the Gaussian curvature K is $-k^2$, we have

$$\widetilde{M}^2 = k^2\left(1 - \widetilde{F}^2\right), \qquad \text{that is,} \qquad \widetilde{F}^2 + \left(\frac{\widetilde{M}}{k}\right)^2 = 1.$$

So there exists a function θ such that

$$\widetilde{F} = \cos\theta, \qquad \widetilde{M} = k\sin\theta.$$

Since the surface is regular, $1 - \widetilde{F}^2 = 1 - \cos^2\theta > 0$ holds. So θ can move on the interval $(0, \pi)$ or $(\pi, 2\pi)$. In the latter case, replacing η by $-\eta$ and θ by $\pi - \theta$, we have the conclusion. □

Remark 3.10. The parameter (ξ, η) as in (3.4) is called the *asymptotic Chebyshev net*.

Example 3.11. $p(u, v) := \bigl(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v\bigr)$.

Exercises

3-1 Let a and b be real numbers with $a \neq 0$ and

$$p(u, v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0, 0, u).$$

Find a coordinate change $(u, v) \mapsto (\xi, \eta)$ to an asymptotic Chebyshev net for p, and give an explicit expression of θ as a function in (ξ, η).

3-2 Let (ξ, η) be an asymptotic Chebyshev net (3.4) on a surface. Assume another parameter (x, y) is also an asymptotic Chebyshev net. Prove that (x, y) satisfies

$$(x, y) = (\pm\xi + x_0, \pm\eta + y_0) \qquad \text{or} \qquad (x, y) = (\pm\eta + x_0, \pm\xi + y_0)$$

where x_0 and y_0 are constants.

[2] Advanced Topics of Geometry A1