4 A construction of pseudospherical surfaces

4.1 Gauss-Weingarten equation

Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a pseudospherical surface of constant Gaussian curvature -1 defined on a domain $U \subset \mathbb{R}^2$. By the result of the previous lecture, we may assume the coordinate system (x, y) on U is the asymptotic Chebyshev net:

$$ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2, \qquad I\!I = 2\sin\theta\, dx\, dy, \tag{4.1}$$

where $\theta = \theta(x, y)$ is a smooth function in (x, y) valued on an interval $(0, \pi)$. Now we define a new coordinate system (u, v) by

$$x = \frac{1}{2}(u - v), \qquad y = \frac{1}{2}(u + v), \tag{4.2}$$

and denote the new parametrization $p((u - v)/2, (u + v)/2)$ by $p(u, v)$. Then the first and second fundamental forms are written as

$$ds^2 = \cos^2\frac{\theta}{2}\, du^2 + \sin^2\frac{\theta}{2}\, dv^2, \qquad I\!I = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2). \tag{4.3}$$

Since $|p_u| = \cos\frac{\theta}{2}$, $|p_v| = \sin\frac{\theta}{2}$, and p_u is perpendicular to p_v, we can take the orthornomal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ satisfying

$$p_u = \cos\frac{\theta}{2}\boldsymbol{e}_1, \quad p_v = \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad \nu = \boldsymbol{e}_3, \tag{4.4}$$

where ν is the unit normal vector field of p. So we get the map

$$\mathcal{F} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)\colon U \longrightarrow \mathrm{SO}(3)$$

called the *frame* or an *adapted frame* of the surface, here $\mathrm{SO}(3)$ is the set of 3×3-orthogonal matrices with positive determinant. The following formula is a consequence of the Gauss-Weingarten equation (cf. Theorem 4.2 in MTH.B405, see also Exercise 4-2 in the same class).

Proposition 4.1. *Under the situation above, the frame $\mathcal{F}$ satisfies*

$$\begin{cases} \mathcal{F}_u &= \mathcal{F}\Omega \\ \mathcal{F}_v &= \mathcal{F}\Lambda \end{cases}; \qquad \Omega = \begin{pmatrix} 0 & -\theta_v/2 & -\sin\frac{\theta}{2} \\ \theta_v/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -\theta_u/2 & 0 \\ \theta_u/2 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

Moreover, the function $\theta = \theta(u, v)$ satisfies the sine-Gordon equation

$$\theta_{uu} - \theta_{vv} = \sin\theta. \tag{4.5}$$

Proof. In spite of the direct conclusion of the Gauss-Weingarten equation, we'll give a direct proof for a sake of convenience. Differentiating the first equality of (4.4) in u, we have

$$p_{uu} = -\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \cos\frac{\theta}{2}(\boldsymbol{e}_1)_u, \tag{4.6}$$

$$p_{uu}\cdot\boldsymbol{e}_2 = \cos\frac{\theta}{2}\big((\boldsymbol{e}_1)_u\big)\cdot\boldsymbol{e}_2, \tag{4.7}$$

$$p_{uu}\cdot\boldsymbol{e}_3 = \cos\frac{\theta}{2}\big((\boldsymbol{e}_1)_u\big)\cdot\boldsymbol{e}_3 \tag{4.8}$$

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where the third equality is nothing but the definition of the second fundamental form. On the other hand, by the definition of the first and second fundamental forms, we have

$$\sin\frac{\theta}{2}\, p_{uu}\cdot \boldsymbol{e}_2 = p_{uu}\cdot p_v = (p_u\cdot p_v)_u - p_u\cdot p_{uv} = -\frac{1}{2}(p_u\cdot p_u)_v = \frac{\theta_v}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \tag{4.9}$$

$$p_{uu}\cdot \boldsymbol{e}_3 = p_{uu}\cdot \nu = \cos\frac{\theta}{2}\sin\frac{\theta}{2}. \tag{4.10}$$

Since $(\boldsymbol{e}_1)_u\cdot \boldsymbol{e}_1 = \frac{1}{2}(\boldsymbol{e}_1\cdot\boldsymbol{e}_1)_u = 0$, we have

$$\left(\boldsymbol{e}_1\right)_u = \frac{\theta_v}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}\boldsymbol{e}_3,$$

which proves the first column of Ω. On the other hand,

$$0 = p_{vu}\cdot \boldsymbol{e}_3 = \left(\sin\frac{\theta}{2}\boldsymbol{e}_2\right)_u\cdot \boldsymbol{e}_3 = \sin\frac{\theta}{2}\left(\boldsymbol{e}_2\right)_u\cdot\boldsymbol{e}_3,$$

proving the $(3,2)$-component of Ω. Since $\mathcal{F}$ is orthogonal, Ω is skew-symmetric. Thus we get the expression of Ω. The components of Λ are obtained in the similar way. □

Remark 4.2. The equation (4.5) is equivalent to the equation

$$\theta_{xy} = \sin\theta, \tag{4.11}$$

which is the integrability condition with respect to the asymptotic Chebyshev net.

As a converse assertion, the fundamental theorem for surface theory deduces

Theorem 4.3. *Let $\theta\colon U\to(0,\pi)$ be a smooth function defined on a simply connected domain $U\subset\mathbb{R}^2$ satisfying the sine-Gordon equation (4.5) Then there exists a regular parametrization $p\colon U\to\mathbb{R}^3$ of a pseudospherical surface whose first and second fundamental forms are written as (4.3).*

Example

As an example of Theorem 4.3, we construct the surfaces of revolution (cf. Exercise 2-1).

Sine-Gordon equation and the equation of pendulum: We assume the function $\theta=\theta(u,v)$ depends only on the variable v: $\theta=\theta(v)$. Then the sine-Gordon equation turns to be

$$\ddot\theta = -\sin\theta \qquad \left(\dot{} = \frac{d}{dv}\right), \tag{4.12}$$

which is the equation of the motion of pendulums. In particular,

$$\left(\frac{\dot\theta}{2}\right)^2 + \sin^2\frac{\theta}{2} = E^2 \tag{4.13}$$

holds, where E is a non-negative constant. When $E=0$, $\sin(\theta/2)$ must be zero, which does not satisfy $\theta\in(0,\pi)$. On the other hand, when $\underline{E=1}$, the solution is written in an elementary function:

$$\theta = \theta_1 := 4\tan^{-1}\frac{e^v-1}{e^v+1} = 4\tan^{-1}\tanh\frac{v}{2} \tag{4.14}$$

Solving Gauss-Weingarten equation: In our case, the Gauss-Weingarten equation (Proposition 4.1) is rewritten as

$$\begin{cases} \mathcal{F}_u &= \mathcal{F}\varOmega \\ \mathcal{F}_v &= \mathcal{F}\varLambda \end{cases}; \qquad \varOmega = \begin{pmatrix} 0 & -\dot\theta/2 & -\sin\frac{\theta}{2} \\ \dot\theta/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix}, \quad \varLambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}. \tag{4.15}$$

Let

$$c = c(v) := \frac{\dot\theta(v)}{2E}, \qquad s = s(v) := \frac{1}{E}\sin\frac{\theta(v)}{2}. \tag{4.16}$$

Then by (4.13) and (4.12), it holds that

$$c^2 + s^2 = 1, \qquad \dot c = -\cos\frac{\theta}{2}s, \qquad \dot s = \cos\frac{\theta}{2}c. \tag{4.17}$$

Using these, we set the orthogonal matrix $P = P(v)$ by

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}. \tag{4.18}$$

Note that the third column of P is the 0-eigenvector of $\varOmega$. Since

$$\widetilde{\varOmega} := P^{-1}\varOmega P = P^T\varOmega P = E\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and both $\varOmega$ and P are functions depending only on v, the first equation of (4.15) is reduced to

$$(\mathcal{F}P)_u = (\mathcal{F}P)\widetilde{\varOmega},$$

which can be solved as

$$\mathcal{F}P = F_0(v)R(u), \qquad R(u) := \begin{pmatrix} \cos Eu & -\sin Eu & 0 \\ \sin Eu & \cos Eu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where F_0 is an SO(3)-valued function in v. Substituting this into the second equation of (4.15),

$$\begin{aligned} \dot F_0 &= (\mathcal{F}PR^T)_v\mathcal{F}_vPR^T + \mathcal{F}\dot PR^T = \mathcal{F}\varLambda PR^T + \mathcal{F}\dot PR^T \\ &= F_0RP^T\varLambda PR^T + F_0RP^T\dot PR^T = F_0R\left(P^T\varLambda P + P^T\dot P\right)R^T = O \end{aligned}$$

holds because of (4.17) and the definition of $\varLambda$. Hence $F_0(v)$ is constant, and by choosing an appropriate initial condition, we obtain

$$\mathcal{F} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = R(u)P(v). \tag{4.19}$$

Hence we have

$$\boldsymbol{e}_1 = \begin{pmatrix} \cos Eu \\ \sin Eu \\ 0 \end{pmatrix} = \boldsymbol{u}_1, \qquad \boldsymbol{e}_2 = c(v)\begin{pmatrix} -\sin Eu \\ \cos Eu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ s(v) \end{pmatrix} = \frac{\dot\theta}{2E}\boldsymbol{u}_2 + \frac{1}{E}\sin\frac{\theta}{2}\boldsymbol{u}_3,$$

where $R = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$. By (4.4), the corresponding surface $p = p(u,v)$ satisfies

$$dp = \cos\frac{\theta(v)}{2}\boldsymbol{v}_1(u)\,du + \frac{\dot\theta(v)}{2E}\sin\frac{\theta(v)}{2}\boldsymbol{v}_2(u)\,dv + \frac{1}{E}\sin^2\frac{\theta(v)}{2}\boldsymbol{v}_3\,dv. \tag{4.20}$$

Integrating this, we obtain

$$p = \frac{-1}{E}\cos\frac{\theta}{2}\boldsymbol{v}_2 + \frac{1}{E}\boldsymbol{v}_3\int_{v_0}^{v}\sin^2\frac{\theta(t)}{2}\,dt,$$

which is a surface of revolution.

Exercises

4-1 The constant function $\theta(u,v)=0$ is a solution of the sine-Gordon equation (4.5) although it does not satisfy the condition $0<\theta<\pi$. In this case, explain what happens on the solution of the equation in Proposition 4.1 and resulting "surface" $p(u,v)$.

4-2 Let $\theta=\theta(x,y)$ be a solution of the sine-Gordon equation $\theta_{xy}=\sin\theta$. Assume a function φ satisfies

$$\left(\frac{\varphi-\theta}{2}\right)_x = a\sin\frac{\varphi+\theta}{2}, \qquad \left(\frac{\varphi+\theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi-\theta}{2},$$

where a is a non-zero constant. Prove that φ is also a solution of the sine-Gordon equation.