5 Hilbert's theorem

Completeness of Riemannian manifolds

Riemannian manifolds. A *Riemannian manifold* is a manifold equipped with a *Riemannian metric* ds^2, that is, for each point $p \in M$, $(ds^2)_\mathrm{P}$ gives a (positive definite) inner product of the tangent space $T_\mathrm{P}M$ of M at P, and $\mathrm{P} \mapsto (ds^2)_\mathrm{P}$ satisfies an appropriate smoothness condition[3].

Example 5.1. Let $\langle\ ,\ \rangle$ be the canonical inner product of $\mathbb{R}^n$. Identifying $T_\mathrm{P}\mathbb{R}^n$ with $\mathbb{R}^n$, $\langle\ ,\ \rangle$ induces the inner product on the tangent space. Thus, $\mathbb{R}^n$ is regarded as a Riemannian manifold with $ds^2 = \langle\ ,\ \rangle$, which is called the *Euclidean space*.

Let M be a 2-dimensional manifold and $p\colon M \to \mathbb{R}^3$ a smooth map[4]. A map p is said to be an *immersion* if the rank of the differential $(dp)_\mathrm{P}$ is 2 for each $\mathrm{P} \in M$. If this is the case, setting

$$(ds^2)_\mathrm{P}(X_\mathrm{P}, Y_\mathrm{P}) := \langle dp_\mathrm{P}(X_\mathrm{P}), dp_\mathrm{P}(Y_\mathrm{P})\rangle \qquad X_\mathrm{P}, Y_\mathrm{P} \in T_\mathrm{P}M$$

we obtain the Riemannian metric ds^2 on M, which is called the *induced metric* of $\langle\ ,\ \rangle$ by p.

In a local coordinate system $\big(U, (u,v)\big)$ on M, the map p is considered as an $\mathbb{R}^3$-valued function in variables (u,v). Then p is an immersion if and only if the derivatives p_u and p_v are linearly independent at each point on U, that is nothing but the condition for regular parametrization. By using standard notation of manifold theory, p_u and p_v are expressed as

$$p_u = dp\left(\frac{\partial}{\partial u}\right), \qquad p_v = dp\left(\frac{\partial}{\partial v}\right).$$

Hence the induced metric ds^2 is determined by its components

$$E := ds^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \left\langle dp\left(\frac{\partial}{\partial u}\right), dp\left(\frac{\partial}{\partial u}\right)\right\rangle = \langle p_u, p_u\rangle,$$
$$F := ds^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \langle p_u, p_v\rangle, \quad G := ds^2\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \langle p_v, p_v\rangle.$$

In this sense, we write the induced metric as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

which is the *first fundamental form* of the surface p.

Completeness. Let $\gamma\colon [a,b] \to M$ be a C^∞-curve and $\gamma'(t) \in T_{\gamma(t)}M$ its *velocity vector* at $\gamma(t)$.

Definition 5.2. The length of the curve γ (with respect to ds^2) is defined by

$$\mathcal{L}_{ds^2}(\gamma) := \int_a^b |\gamma'(t)|\,dt, \qquad \text{where} \qquad |\gamma'(t)| = \sqrt{ds^2(\gamma'(t), \gamma'(t))}.$$

For two points P, Q $\in M$, we denote $\mathcal{C}_{\mathrm{P},\mathrm{Q}}$ the set of smooth curves on M joining P and Q. Then it is known (cf. [dC92, Prop. 2.5 in Chap. 7]) that if we define

$$d_{ds^2}(\mathrm{P}, \mathrm{Q}) := \inf\left\{\mathcal{L}_{ds^2}(\gamma)\,;\, \gamma \in \mathcal{C}_{\mathrm{P},\mathrm{Q}}\right\},$$

the function $d_{ds^2}\colon M \times M \to \mathbb{R}$ satisfies the axiom of distance. Moreover, the topology of M induced by the distance d_{ds^2} coincides with the original topology of M.

We call d_{ds^2} the *distance* with respect to the metric ds^2.

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[3] "For any smooth vector fields X, Y on M, the funtion $M \ni \mathrm{P} \mapsto (ds^2)_\mathrm{P}(X_\mathrm{P}, Y_\mathrm{P})$ is smooth."

[4] Although we restrict our arguments into 2-dimensional case, the contents here are valid for higher dimensions.

Definition 5.3. A Riemannian manifold M with a Riemannian metric ds^2 is said to be *complete* if the distance function d_{ds^2} is complete, in other words, any Cauchy sequence with respect to d_{ds^2} converges in M.

Remark 5.4. Completeness is equivalent to each of the following conditions (*Hopf-Rinow Theorem*, cf. [dC92, Chap. 7]): (1) Any geodesic is defined whole on $\mathbb{R}$. (2) Any divergent path has infinity length. (3) Any bounded closed subset is compact.

Example 5.5. The Euclidean space $\mathbb{R}^n$ is complete. In fact, induced distance is the usual distance function. On the other hand, $M := \mathbb{R}^2 \setminus \{\mathrm{O}\}$ with the canonical metric of $\mathbb{R}^2$ is incomplete. In fact, the Cauchy sequence $\{(1/n, 0, \dots, 0)\}$ on M does not converge to any point in M.

The hyperbolic plane. The *hyperbolic plane* is the Riemannian manifold (H^2, ds^2), where $H^2 = \{(x, y)\,;\, y > 0\}$ is the upper-half plane and $ds^2 = (dx^2 + dy^2)/y^2$ (cf. Corollary 1.13), is called the *hyperbolic plane*. As seen in Section 1, this is a model of the non-Euclidean geometry.

Proposition 5.6. *The hyperbolic plane is complete.*

Proof. First, we compute the distance of two points aligned on the vertical line: Let $\mathrm{P} := (x_1, y_1)$, $\mathrm{Q} := (x_2, y_2) \in H^2$. For a curve $\gamma(t) = (x(t), y(t))$ $(0 \leqq t \leqq 1)$ in $\mathcal{C}_{\mathrm{P,Q}}$,

$$\mathcal{L}_{ds^2}(\gamma) = \int_0^1 \frac{1}{y(t)}\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\,dt \geqq \int_0^1 \frac{1}{y(t)}|\dot{y}(t)|\,dt \geqq \left|\int_0^1 \frac{\dot{y}(t)}{y(t)}\,dt\right| = \bigl|\log y(t)\bigr|\Big|_0^1 = \left|\log\frac{y_2}{y_1}\right|.$$

Then

$$d_{ds^2}(\mathrm{P}, \mathrm{Q}) \geqq \left|\log\frac{y_2}{y_1}\right| \qquad \text{for} \quad \mathrm{P} = (x_1, y_1), \quad \mathrm{Q} = (x_2, y_2).$$

Let $\{\mathrm{P}_n = (x_n, y_n)\}$ be a Cauchy sequence with respect to d_{ds^2} and fix a point $\mathrm{O} = (0, 1)$. Since a Cauchy sequence is bounded, there exists a positive number m such that

$$m \geqq d_{ds^2}(\mathrm{O}, \mathrm{P}_n) \geqq |\log y_n|$$

holds for all n, that is $\{\log y_n\}$ is bounded. Hence there exists positive constants a and b such that $a < y_n < b$ for all n. Here, on the domain $U := \{(x, y) \in H^2\,;\, a < y < b\}$,

$$\frac{1}{b^2}(dx^2 + dy^2) \leqq ds^2 \leqq \frac{1}{a^2}(dx^2 + dy^2).$$

Hence the sequence $\{\mathrm{P}_n\}$ on U is a Cauchy sequence with respect to the Euclidean metric, and then it converges to a point on the closure $\overline{U}$ of U. □

Hilbert's theorem

The purpose of this section is to give the following theorem

Theorem 5.7 (D. Hilbert (1901) [Hil01])**.** *There exists no complete pseudospherical surface.*

Existence of a global asymptotic Chebyshev net. Let $p\colon M \to \mathbb{R}^3$ be a *complete* immersion of 2-dimensional manifold M to $\mathbb{R}^3$, and denote by ds^2 the induced metric.

Proposition 5.8. *There exists a smooth function $\theta\colon \mathbb{R}^2 \to (0, \pi)$ and a smooth map $\pi\colon \mathbb{R}^2 \to M$ such that $\tilde{p} := p \circ \pi\colon \mathbb{R}^2 \to \mathbb{R}^3$ is an immersion with the first and second fundamental forms as*

$$ds^2 = dx^2 + 2\cos\theta\,dx\,dy + dy^2, \qquad I\!I = 2\sin\theta\,dx\,dy, \qquad \theta_{xy} = \sin\theta, \qquad 0 < \theta < \pi. \tag{5.1}$$

Proof. By the existence of the asymptotic Chebyshev net (Theorem 3.9) implies, there exists a domain $U_0 \subset \mathbb{R}^2$ and a map $\pi\colon U_0 \to M$ such that the coordinates (x, y) of $\mathbb{R}^2$ gives the asymptotic Chebyshev net of the surface, that is, (5.1) holds on U_0. We let $\mathcal{U}$ be the set of $U \subset \mathbb{R}^2$ containing U_0 satisfying (5.1), and take the maximal element U of the semi-ordered set $\mathcal{U}$ with respect to the inclusion.

Assume $U \subsetneq \mathbb{R}^2$. Since $\partial U := \overline{U} \setminus U$ is not empty, we can take $\mathrm{P} = (x_0, y_0) \in \partial U$. Let $\{(x_n, y_n)\}$ a sequence in U converging to (x_0, y_0). Then both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers, and since

$$ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2 \leqq 2(dx^2 + dy^2),$$

$\{(x_n, y_n)\}$ is a Cauchy sequence with respect to the metric ds^2. Thus, setting $\mathrm{P}_n := \pi(x_n, y_n)$, $\{\mathrm{P}_n\}$ is a Cauchy sequence, and by completeness, it converges to $\mathrm{P} \in M$. Using Theorem 3.9 again, there exists an asymptotic Chebyshev net $(\hat{x}, \hat{y})$ on a neighborhood W of P. Since $\pi(U) \cap W \neq \varnothing$, the parameter $(\hat{x}, \hat{y})$ is related to (x, y) as $(\hat{x}, \hat{y}) = (\pm x + a, \pm y + b)$ or $(\pm y + a, \pm x + b)$, because of Exercise 3-2. Then by a coordinate change, we may assume $(\hat{x}, \hat{y})$ coincides with (x, y) without loss of generality. Hence the asymptotic Chebyshev net (x, y) can be extended across P, which contradicts to the maximality of U. □

Proof of Hilbert's theorem. Let θ be a smooth function on $\mathbb{R}^2$ satisfying

$$\theta_{xy} = \sin\theta, \qquad 0 < \theta < \pi. \tag{5.2}$$

In particular, $\sin\theta > 0$ holds on $\mathbb{R}^2$.

Lemma 5.9. *Let x_1, x_2, y_1, y_2 be real numbers with $x_1 < x_2$ and $y_1 < y_2$. Then*

$$\theta(x_2, y_2) - \theta(x_1, y_2) = \theta(x_2, y_1) - \theta(x_1, y_1) + \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} dx \sin\theta(x, y). \tag{5.3}$$

Proof. Letting $R := [x_1, x_2] \times [y_1, y_2]$, we have

$$\begin{aligned} \iint_R \sin\theta(x, y)\, dx\, dy &= \iint_R \theta_{xy}(x, y)\, dx\, dy = \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} (\theta_y)_x(x, y)\, dx \\ &= \int_{y_1}^{y_2} \left(\theta_y(x_2, y) - \theta_y(x_1, y)\right) dy = \theta(x_2, y_2) - \theta(x_2, y_1) - \theta(x_1, y_2) + \theta(x_1, y_1). \end{aligned}$$ □

Lemma 5.10. *There exists $y_1 \in \mathbb{R}$ such that the function $x \mapsto \theta(x, y_1)$ is non-constant.*

Proof. If $x \mapsto \theta(x, y_0)$ is constant, (5.3) implies

$$\theta(x_1, y_1) - \theta(x_0, y_1) = \theta(x_1, y_0) - \theta(x_0, y_0) + \int_{y_0}^{y_1} \int_{x_0}^{x_1} \sin\theta\, dx\, dy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} \sin\theta\, dx\, dy > 0,$$

where $y_1 > y_0$ and $x_1 > x_0$. Thus $x \mapsto \theta(x, y_1)$ is strictly increasing. □

Let $y_1 \in \mathbb{R}$ as in Lemma 5.10. Then there exists an interval $I := [x_A, x_B]$ such that θ is strictly increasing or decreasing. Then, by a coordinate change $(x, y) \mapsto (-x, -y)$ and a translation on the xy-plane, we may assume without loss of generality that

$$x \mapsto \theta(x, 0) \quad \text{is strictly increasing on } [0, x_1] \qquad (x_1 > 0). \tag{5.4}$$

Lemma 5.11. *Under the situation above, $x \mapsto \theta(x, y_1)$ is strictly increasing on $[0, x_1]$ if $y_1 \geqq 0$.*

Proof. Apply Lemma 5.9 for $[x, x'] \times [0, y_1]$ where $0 \leqq x < x' \leqq x_1$. □

Take real numbers x_2 and x_3 satisfying $0 < x_3 < x_2 < x_1$, and let

$$\varepsilon := \theta(x_1, 0) - \theta(x_2, 0), \qquad \varepsilon' := \theta(x_3, 0) - \theta(0, 0) > 0,$$

Lemma 5.12. *There exists* $(x_0, y_0) \in (x_3, x_2) \times (0, \infty)$ *such that*

$$\theta(x_0, y_0) > \pi - \frac{\varepsilon}{2}.$$

Proof. Assume $\theta(x, y) \leqq \pi - \varepsilon/2$ holds on $R := [x_3, x_2] \times [0, \infty)$. Then Lemmas 5.9 and 5.11 yield

$$\pi - \frac{\varepsilon}{2} \geqq \theta(x, y) > \theta(x, y) - \theta(0, y) \geqq \theta(x, 0) - \theta(0, 0) \geqq \theta(x_3, 0) - \theta(0, 0) = \varepsilon'$$

on R. Hence $\sin\theta \geqq \delta$ holds on R, where $\delta := \min\{\sin(\pi - \varepsilon/2), \sin\varepsilon'\} > 0$. So, for $(x, y) \in R$,

$$\begin{aligned}\theta(x, y) > \theta(x, y) - \theta(x_3, y) &= \theta(x, 0) - \theta(x_3, 0) + \int_0^y \int_{x_3}^x \sin\theta \, dx \, dy \\ &\geqq \theta(x, 0) - \theta(x_3, 0) + (x_3 - x)y\delta.\end{aligned}$$

Letting $y \to +\infty$, the right-hand side diverges to $+\infty$. Then $\theta(x, y)$ exceeds π, a contradiction. □

Proof of Theorem 5.7. Let $p\colon M \to \mathbb{R}^3$ be a complete pseudospherical surface. Then by Proposition 5.8, there exists a global asymptotic Chebyshev net. In particular, there exists a function $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying (5.2).

Under the situation in this subsection, we have

$$\begin{aligned}\theta(x_1, y_0) - \theta(x_0, y_0) \geqq \theta(x_1, y_0) - \theta(x_2, y_0) &= \theta(x_1, 0) - \theta(x_2, 0) + \int_0^{y_0} dy \int_{x_2}^{x_1} \sin\theta \, dx \, dy \\ &\geqq \theta(x_1, 0) - \theta(x_2, 0) = \varepsilon.\end{aligned}$$

Hence

$$\theta(x_1, y_0) \geqq \theta(x_0, y_0) + \varepsilon \geqq \pi + \frac{\varepsilon}{2},$$

contradicting the assumption $\theta \in (0, \pi)$. □

Exercises

5-1 Consider a map

$$p\colon \mathbb{R}^2 \ni (u, v) \longmapsto (v \cosh u, v, v \sinh u) \in \mathbb{R}^3.$$

(1) Verify that the image $p(\mathbb{R}^2)$ is contained in the cone $\{(x, y, z) \in \mathbb{R}^3\,;\, x^2 - y^2 - z^2 = 0\}$.

(2) Is the induced metric $p^* \langle\ ,\ \rangle$ complete on $\mathbb{R}^2$?

5-2 Prove that the shortest curve (with respect to the canonical Riemannian metric) joining $\mathrm{O} := (0, 0)$ and $\mathrm{P} := (L, 0)$ $(L > 0)$ on the Euclidean plane is the line segment joining them. (Hint: For a curve $\gamma(t) = \bigl(x(t), y(t)\bigr)$ $(0 \leqq t \leqq 1)$ joining O and P, and apply the inequality similar to the proof of Proposition 5.6, and consider the equality conditions.)