6 Lorentz-Minkowski space

The Lorentz inner product

The *Lorentz inner product* on the vector space $\mathbb{R}^{n+1}$ is a non-degenerate bilinear form

$$\text{(6.1)}\quad \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\ni(\boldsymbol{x},\boldsymbol{y})\mapsto\langle\boldsymbol{x},\boldsymbol{y}\rangle := -x_0y_0+x_1y_1+\dots+x_ny_n=\boldsymbol{x}^TY\boldsymbol{y},$$
$$\text{where}\quad \boldsymbol{x}=\begin{pmatrix}x_0\\x_1\\\vdots\\x_n\end{pmatrix},\quad \boldsymbol{y}=\begin{pmatrix}y_0\\y_1\\\vdots\\y_n\end{pmatrix},\quad\text{and}\quad Y:=\begin{pmatrix}-1&0&\dots&0\\0&1&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&1\end{pmatrix}.$$

The *Lorentz group* is the group consists of the linear transformations of $\mathbb{R}^{n+1}$ preserving the Lorentz inner product $\langle\ ,\ \rangle$. We denote it by $\mathrm{O}(n,1)$:[5]

$$\text{(6.2)}\qquad \begin{aligned}\mathrm{O}(n,1)&=\{A\in\mathrm{M}_{n+1}(\mathbb{R}),;\ \langle A\boldsymbol{x},A\boldsymbol{y}\rangle=\langle\boldsymbol{x},\boldsymbol{y}\rangle\ \text{ for any } \boldsymbol{x},\ \boldsymbol{y}\in\mathbb{R}^{n+1}\}\\&=\{A\in\mathrm{M}_{n+1}(\mathbb{R})\,;\,A^TYA=Y\},\end{aligned}$$

where $\mathrm{M}_{n+1}(\mathbb{R})$ is the set of $(n+1)\times(n+1)$-real matrices with real components.

Lemma 6.1. *Let* $A=(a_{ij})_{i,j=0,\dots,n}\in\mathrm{O}(n+1,1)$. *Then* $\det A=\pm1$ *and* $|a_{00}|\geqq 1$.

Proof. By (6.2), $(\det A)^2=1$, that is, $\det A=\pm1$. On the other hand, letting $\boldsymbol{e}_0:=(1,0,\dots,0)^T$,

$$\langle A\boldsymbol{e}_0,A\boldsymbol{e}_0\rangle=-(a_{00})^2+(a_{10})^2+\dots+(a_{n0})^2,\qquad\text{and}\qquad\langle\boldsymbol{e}_0,\boldsymbol{e}_0\rangle=-1$$

hold. Since these two values are equal by (6.2), we have the second assertion. □

Introducing the topology $\mathrm{M}_{n+1}(\mathbb{R})$ by the identification with $\mathbb{R}^{2(n+1)}$, we know that [6]

Fact 6.2. *The set* $\mathrm{O}(n,1)\subset\mathrm{M}_{n+1}(\mathbb{R})$ *consits of for connected components,*

$$\begin{aligned}\mathrm{SO}_+(n,1):=&\{A=(a_{ij})_{i,j=0,\dots,n}\in\mathrm{M}_{n+1}(\mathbb{R})\,;\det A>0,a_{00}>0\},\\&\{A=(a_{ij})_{i,j=0,\dots,n}\in\mathrm{M}_{n+1}(\mathbb{R})\,;\det A<0,a_{00}>0\},\\&\{A=(a_{ij})_{i,j=0,\dots,n}\in\mathrm{M}_{n+1}(\mathbb{R})\,;\det A>0,a_{00}<0\},\\&\{A=(a_{ij})_{i,j=0,\dots,n}\in\mathrm{M}_{n+1}(\mathbb{R})\,;\det A<0,a_{00}<0\}.\end{aligned}$$

Definition 6.3. A vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}$ is said to be

- *space-like* if either $\langle\boldsymbol{x},\boldsymbol{x}\rangle>0$ or $\boldsymbol{x}=\boldsymbol{0}$,
- *time-like* if $\langle\boldsymbol{x},\boldsymbol{x}\rangle<0$, and
- *light-like* or *null* or *isotropic* if $\langle\boldsymbol{x},\boldsymbol{x}\rangle=0$ and $\boldsymbol{x}\neq\boldsymbol{0}$.

The above properties of vectors are called *causality*.[7]

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[5] The symbol is written in various ways, for example $\mathrm{O}(n,1)$ instead of $\mathrm{O}(n+1,1)$.

[6] We omit the proof.

[7] The words "time-like", "space-like", "light-like" and "causality" come from the special relativity, which is one of the important applications of Lorentzian geometry. See Example 6.7.

Example 6.4. Let $\{\boldsymbol{e}_0, \dots, \boldsymbol{e}_n\}$ be the canonical basis of $\mathbb{R}^{n+1}$:

$$\boldsymbol{e}_0 := (1, 0, \dots, 0)^T, \quad \boldsymbol{e}_1 := (0, 1, \dots, 0)^T, \quad \dots, \quad \boldsymbol{e}_n := (0, 0, \dots, 1)^T,$$

which is *orthonormal* with respect to $\langle\ ,\ \rangle$:

$$|\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle| = \delta_{ij},$$

where δ_{ij} is the Kronecker's delta symbol. The first vector $\boldsymbol{e}_0$ is time-like and others are space-like. Moreover, the subspace $\operatorname{Span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$, which is the orthogonal complement of $\boldsymbol{e}_0$, consists of space-like vectors.

Lemma 6.5. *Let $\boldsymbol{x} \in \mathbb{R}^{n+1}$ be a time-like vector. Then its orthogonal complement*

$$\boldsymbol{x}^\perp := \{\boldsymbol{y} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{y}, \boldsymbol{x} \rangle = 0\}$$

is an n-dimensional linear subspace of $\mathbb{R}^{n+1}$, consisting of space-like vectors.

Proof. Since $\boldsymbol{x}^\perp$ is the kernel of the linear map $\varphi : \mathbb{R}^{n+1} \ni \boldsymbol{y} \mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{R}$, it is a liner subspace of $\mathbb{R}^{n+1}$. Moreover, $\varphi(\boldsymbol{x}) < 0$ implies that the image of φ is $\mathbb{R}$. Hence $\dim \operatorname{Ker} \varphi = n$. Denote by $\boldsymbol{x} = (x_0, x_1, \dots, x_n)^T$, which is time-like:

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = -(x_0)^2 + \sum_{j=1}^{n} (x_j)^2 < 0 \qquad \text{that is,} \qquad \sum_{j=1}^{n} x_j^2 < (x_0)^2.$$

Since $\boldsymbol{y} = (y_0, y_1, \dots, y_n)^T \in \boldsymbol{x}^\perp$ satisfies $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{j=1}^{n} x_j y_j = 0$,

$$(x_0 y_0)^2 = \left(\sum_{j=1}^{n} x_j y_j \right)^2 \leqq \left(\sum_{j=1}^{n} x_j^2 \right) \left(\sum_{j=1}^{n} y_j^2 \right) < (x_0)^2 \left(\sum_{j=1}^{n} y_j^2 \right).$$

Hence $\langle \boldsymbol{y}, \boldsymbol{y} \rangle > 0$, if $\boldsymbol{y} \neq \boldsymbol{0}$. □

The Lorentz-Minkowski space

The $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$ is the $(n+1)$-manifold $\mathbb{R}^{n+1}$ whose tangent space $T_{\mathrm{P}}\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$ is endowed with Lorentzian metric $\langle\ ,\ \rangle$.

Fact 6.6. *An isometry of $\mathbb{L}^{n+1}$ is in the form*

$$\mathbb{L}^{n+1} \ni \boldsymbol{x} \longmapsto A\boldsymbol{x} + \boldsymbol{a} \in \mathbb{L}^{n+1}, \qquad A \in \mathrm{O}(n+1, 1), \quad \boldsymbol{a} \in \mathbb{R}^{n+1}.$$

Example 6.7 (The special relativity)**.** The *special relativity* is a geometry of the LorentzMinkowski 4-space $\mathbb{L}^4$ as follows: Consider $\mathbb{L}^4$ as a *space-time*, where $(x, y, z) = (x_1, x_2, x_3)$ is the position vector of the space $\mathbb{R}^3$ and $t = x_0/c$ is the "time", where c is the positive constant called the *speed of light in vacuum*. In fact, for a light-like vector $\boldsymbol{v}$, a curve $\gamma(s) := \boldsymbol{x} + s\boldsymbol{v}$, called the *light-like line*, represents a motion of a particle in the speed of light. A Lorentz transformation of this curve also represents a motion in light speed, which is interpreted as the *principle of invariance of the speed of light*. More precisely, if a transformation of the space-time preserves light-like lines, the transformation is a Lorentz transformation.

Hyperbolic Space

Let

$$H^n := \{\boldsymbol{x} = (x_0, \dots, x_n)^T \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\} \subset \mathbb{L}^{n+1}$$

Lemma 6.8. *H^n is a connected n-dimensional submanifold of $\mathbb{R}^{n+1}$.*

Proof. Let

$$F\colon \mathbb{R}^{n+1} \ni \boldsymbol{x} = (x_0, \dots, x_n)^T \mapsto \langle \boldsymbol{x}, \boldsymbol{x} \rangle + 1 \in \mathbb{R}$$
$$G\colon \mathbb{R}^{n+1} \ni \boldsymbol{x} = (x_0, \dots, x_n)^T \mapsto x_0 \in \mathbb{R}.$$

Then by the implicit function theorem, $X := F^{-1}(\{0\})$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Here, if $\boldsymbol{x} \in X$, $x_0^2 = 1 + x_1^2 + \dots + x_n^2$ holds, that is, $G|_X = (-\infty, -1] \cup [1, +\infty)$. Since G is continuous, X is disconnected. Moreover, $H^n = F^{-1}(\{0\}) \cap G^{-1}([1, \infty))$ is connected, because it can be expressed as a graph $x_0 = \sqrt{1 + x_1^2 + \dots + x_n^2}$. □

Lemma 6.9. *For each $\boldsymbol{x} \in H^n$,*

(1) *$T_{\boldsymbol{x}} H^n = \boldsymbol{x}^\perp$,*

(2) *and the restriction of $\langle\ ,\ \rangle$ to $T_{\boldsymbol{x}} H^n$ is positive definite.*

Proof. Let $\gamma(t)$ be a curve on H^n with $\gamma(0) = \boldsymbol{x}$. Differentiating $\langle \gamma(t), \gamma(t) \rangle = -1$ in t, we have $\langle \gamma'(t), \gamma(t) \rangle = 0$. In particular, $\gamma'(0) \in T_{\boldsymbol{x}} H^n$ is perpendicular to $\boldsymbol{x}$. Hence $T_{\boldsymbol{x}} H^n \subset \boldsymbol{x}^\perp$. Then Lemma 6.5 yields the both side are n-dimensional. Thus (1) is proven. In addition, the lemma implies (2). □

Thus, by restricting $\langle\ ,\ \rangle$ to the tangent space of H^n, we obtain a Riemannian manifold $(H^n, \langle\ ,\ \rangle)$.

Definition 6.10. The manifold $(H^n, \langle\ ,\ \rangle)$ is called the *hyperbolic space*.

Fact 6.11. *The isometry group of the hyperbolic space is*

$$\mathrm{SO}_+(n, 1) \cup \{A = (a_{ij})\,;\, \det A = -1, a_{00} > 0\}.$$

Fact 6.12. *Let $\boldsymbol{x} \in H^n$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} H^n$ with $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$. Then*

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

is a geodesic *on H^n with $\gamma_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{x}$ and $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(0) = \boldsymbol{v}$. In particular, the hyperbolic space is* complete *because the geodesics $\gamma_{\boldsymbol{x},\boldsymbol{v}}$ are defined whole on $\mathbb{R}$.*

Exercises

6-1 Let

$$A := \begin{pmatrix} \frac{3}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{pmatrix}, \qquad B := \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix},$$

where u and v are real numbers.

(1) Verify that A and B are elements of $\mathrm{SO}_+(3,1)$.

(2) When A is conjugate to B? (Hint: Compute the eigenvalues.)

6-2 Let

$$S^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{x} = 1\},$$

where "$\cdot$" denotes the Euclidean inner product. Then S^n is an n-dimensional submanifold embedded in $\mathbb{R}^{n+1}$, called the *n-sphere*. We denote the tangent space of S^n at $\boldsymbol{x} \in S^n$ by $T_{\boldsymbol{x}}S^n$, and set

$$U_{\boldsymbol{x}}S^n := \{\boldsymbol{v} \in T_{\boldsymbol{x}}S^n\,;\, |\boldsymbol{v}| = 1\}.$$

(1) Show that $T_{\boldsymbol{x}}S^n$ is expressed as

$$T_{\boldsymbol{x}}S^n = \boldsymbol{x}^{\perp} = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \boldsymbol{x}\cdot\boldsymbol{v} = 0\}.$$

(2) Show that the curve

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos t)\boldsymbol{x} + (\sin t)\boldsymbol{v} \qquad \bigl(\boldsymbol{x} \in S^n, \boldsymbol{v} \in U_{\boldsymbol{x}}S^n\bigr)$$

in $\mathbb{R}^{n+1}$ is a curve on S^n with $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x}$ and $\gamma'_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{v}$, where $' = d/dt$.

(3) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two distinct points of S^n with $\boldsymbol{y} \neq -\boldsymbol{x}$. Find $\boldsymbol{v} \in U_{\boldsymbol{x}}S^n$ and $t_0 \in (-\pi,\pi)$ such that $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$. (Hint: orthogonalization)