7 Models of non-Euclidean geometry

In this section, we treat the 2-dimensional hyperbolic space H^2 for the sake of simplicity:

$$H^2 := \{\boldsymbol{x} = (x_0, x_1, x_2)^T \in \mathbb{L}^3\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1, x_0 > 0\}.$$

Almost all discussions here work for general dimensional case. Throughout this section, we denote the canonical basis of $\mathbb{L}^3 (= \mathbb{R}^3)$ by

$$\boldsymbol{e}_0 := (1, 0, 0)^T, \quad \boldsymbol{e}_1 := (0, 1, 0)^T, \quad \boldsymbol{e}_2 := (0, 0, 1)^T.$$

Isometries

Recall that

$$\mathrm{O}(2,1) := \{A \in \mathrm{M}_3(\mathbb{R})\,;\, A^T Y A = Y\}, \qquad \text{where} \quad Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the set of linear isometry of $\mathbb{L}^3 (= \mathbb{R}^3)$ preserving the Lorentz inner product. The connected component of $\mathrm{O}(2,1)$ containing the identity matrix is

$$\mathrm{SO}_+(3,1) := \{A = (a_{ij})_{i,j=0,1,2} \in \mathrm{O}(2,1)\,;\, \det A = 1, a_{00} > 0\}.$$

Lemma 7.1. *Let $A \in \mathrm{SO}_+(2,1)$. Then $A^{-1} = Y A^T Y$. In particular, $A^T \in \mathrm{SO}_+(2,1)$.*

Proof. Let $A = (a_{ij}) \in \mathrm{SO}_+(2,1)$. Since $Y^2 = \mathrm{id}$, $Y A^T Y A = \mathrm{id}$. Hence $A^{-1} = Y A^T Y$, and then $A Y A^T Y = A A^{-1} = \mathrm{id}$. □

Lemma 7.2. *The liner action of $\mathrm{SO}_+(3,1)$ on $\mathbb{L}^3$ preserves H^2.*

Proof. Let $A = (a_{ij}) \in \mathrm{SO}_+(2,1)$ and $\boldsymbol{x} = (x_0, x_1, x_2)^T \in H^2$. Since it preserves the inner product, $\langle A\boldsymbol{x}, A\boldsymbol{x}\rangle = \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1$ for all $\boldsymbol{x} \in H^2$. Then it is sufficient to show the first component of $A\boldsymbol{x}$ is positive. Here , $A Y A^T = Y$ holds by Lemma 7.1. By the top-left component of this identity and the definition of H^2, we have

$$-(a_{00})^2 + (a_{01})^2 + (a_{02})^2 = -1, \quad -(x_0)^2 + (x_1)^2 + (x_2)^2 = -1, \quad a_{00} > 0, \quad \text{and} \quad x_0 > 0.$$

So, the first component of $A\boldsymbol{x}$ is computed as

$$\begin{aligned} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 &= \sqrt{(a_{01})^2 + (a_{02})^2 + 1}\sqrt{(x_1)^2 + (x_{22})^2 + 1} + a_{01}x_1 + a_{02}x_2 \\ &> \sqrt{(a_{01})^2 + (a_{02})^2}\sqrt{(x_1)^2 + (x_{22})^2} + a_{01}x_1 + a_{02}x_2 \geqq 0. \end{aligned}$$

Here, the final inequality comes from the Cauchy-Schwarz inequality. This completes the proof. □

Lemma 7.3. *The action of $\mathrm{SO}_+(2,1)$ on $\mathbb{L}^3$ is isometric.*

Proof. Let $f\colon H^2 \ni \boldsymbol{x} \mapsto A\boldsymbol{x} \in H^2$, where $A \in \mathrm{SO}_+(2,1)$. Take $\boldsymbol{x} \in H^2$ and $\boldsymbol{v} \in T_{\boldsymbol{x}}H^2 = \boldsymbol{x}^\perp$. Then there exists a curve $\gamma(t)$ on H^2 such that $\gamma(0) = \boldsymbol{x}$ and $\gamma'(0) = \boldsymbol{v}$, where $' = d/dt$. Then $df(\boldsymbol{v}) = (f \circ \gamma)'(0) = (A\gamma(t))'|_{t=0} = A\gamma'(0) = A\boldsymbol{v}$. In other words, the differential $df_{\boldsymbol{x}}\colon T_{\boldsymbol{x}}H^2 \to T_{\boldsymbol{x}}H^2$ is the liner action of the matrix A on $T_{\boldsymbol{x}}H^2 \subset \mathbb{L}^3$. Since A preserves the Lorentz inner product, f is an isometry. □

Lemma 7.4. *The group $\mathrm{SO}_+(2,1)$ acts transitively on the unit tangent bundle UH^2 of H^2, where*

$$UH^2 := \bigcup_{\boldsymbol{x} \in H^2} U_{\boldsymbol{x}}H^2, \qquad U_{\boldsymbol{x}}H^2 := \{\boldsymbol{v} \in T_{\boldsymbol{x}}H^2\,;\, \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 1\}.$$

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Proof. The isometry f as in the proof of Lemma 7.3 induces the map $f_*\colon TH^2 \to TH^2$ as

$$f_*\colon TH^2 \ni \boldsymbol{v} \longmapsto df(\boldsymbol{v}) = A\boldsymbol{v} \in TH^2, \qquad \boldsymbol{v} \in T_{\boldsymbol{x}}H^2, \quad df(\boldsymbol{v}) \in T_{f(\boldsymbol{x})}H^2 = T_{A\boldsymbol{x}}H^2.$$

Since the linear action of A preserves the Lorentz inner product, f_* induces the map $UH^2 \to UH^2$.

Take $\boldsymbol{x} \in H^2$ and $\boldsymbol{v} \in U_{\boldsymbol{x}}H^2$ and Let $\boldsymbol{a}_0 = \boldsymbol{x}$, $\boldsymbol{a}_1 = \boldsymbol{v}$ and $\boldsymbol{a}_2 = Y(\boldsymbol{x}\times\boldsymbol{v})$, where "$\times$" denotes the vector product of the vectors of *Euclidean* space $\mathbb{R}^3$. Then $\langle \boldsymbol{a}_i, \boldsymbol{a}_j\rangle = 0$ if $i \neq j$, $\langle \boldsymbol{a}_0, \boldsymbol{a}_0\rangle = -1$, and $\langle \boldsymbol{a}_j, \boldsymbol{a}_j\rangle = 1$ $(j = 1, 2)$. So we have $A := (\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathrm{O}(2, 1)$. Moreover, the top-left component of A is the first component of $\boldsymbol{x}$, which is positive, and $\det A = \boldsymbol{a}_2\cdot(\boldsymbol{a}_0\times\boldsymbol{a}_1) = \boldsymbol{a}_2\cdot Y\boldsymbol{a}_2 = \langle \boldsymbol{a}_2, \boldsymbol{a}_2\rangle = 1$, where "$\cdot$" is the Euclidean inner product. Thus, we have $A \in \mathrm{SO}_+(2, 1)$.

By definition, the liner transformation by the matrix A maps $\boldsymbol{e}_0$ to $\boldsymbol{a}_0 = \boldsymbol{x}$, and $\boldsymbol{e}_1$ to $\boldsymbol{a}_1 = \boldsymbol{v}$. Since the pair $(\boldsymbol{x}, \boldsymbol{v})$ is taken arbitrarily, the conclusion follows □

Hyperbolic plane and the upper-half plane

We have used the symbol H in Section 1 for the upper-half space as a model of non-Euclidean geometry. To avoid confusing in this section, we denote the upper-half plane and its metric as

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2\,;\, y > 0\}, \qquad ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{7.1}$$

In this subsection, we shall explain the relationship between $H^2 \subset \mathbb{L}^3$ and $\mathbb{R}^2_+$.

Lemma 7.5. *If $\boldsymbol{x} = (x_0, x_1, x_2)^T \in H^3$. Then $x_0 + x_1 > 0$ holds.*

Proof. Since $(x_0)^2 - (x_1)^2 = 1 + (x_2)^2$, we have $(x_0 + x_1)(x_0 - x_1) > 0$. So $x_0 + x_1$ does not change sign on H^2 because H^2 is connected, and it is positive at $\boldsymbol{x} = (1, 0, 0)^T \in H^2$. □

Let

$$\pi\colon H^2 \ni (x_0, x_1, x_2) \longmapsto \left(\frac{x_2}{x_0 + x_1}, \frac{1}{x_0 + x_1}\right) \in \mathbb{R}^2_+. \tag{7.2}$$

Lemma 7.6. *The map π is diffeomorphism, and its inverse is expressed as*

$$\pi^{-1}\colon \mathbb{R}_+ \ni (x, y) \longmapsto \left(\frac{1 + x^2 + y^2}{2y}, \frac{1 - x^2 - y^2}{2y}, \frac{x}{y}\right). \tag{7.3}$$

Proof. Set $(x, y) = (x_2, 1)/(x_0 + x_1)$ for $(x_0, x_1, x_2)^T \in \mathbb{L}^2$ and $(x, y) \in \mathbb{R}^2_+$. Then

$$x_0 + x_1 = \frac{1}{y}, \quad x_2 = \frac{x}{y}, \qquad (x_0 - x_1)(x_0 + x_1) - x_2^2 = 1.$$

Hence we have the expression (7.3), and then π is a bijection. □

Proposition 7.7. *The diffeomorphism $\pi\colon H^2 \to \mathbb{R}^2_+$ is an isometry with respect to the metric on H^2 induced from the Lorentzian inner product, and the metric ds^2 on $\mathbb{R}^2_+$ in* (7.1)*.*

Proof. Differentiating $(x_0, x_1, x_2) = \pi^{-1}(x, y)$, we have

$$\begin{aligned}
dx_0 &= \left(\frac{1 + x^2 + y^2}{2y}\right) dx + \left(\frac{-1 - x^2 + y^2}{2y}\right) dy,\\
dx_1 &= \left(\frac{1 - x^2 - y^2}{2y}\right) dx + \left(\frac{1 - x^2 - y^2}{2y}\right) dy,\\
dx_2 &= \frac{1}{y}\, dx - \frac{x}{y^2}\, dy.
\end{aligned}$$

Then the metric on H^2 is expressed as

$$-dx_0^2 + dx_1^2 + dx_2^2 = \frac{dx^2 + dy^2}{y^2} = ds^2.$$

This completes the proof. □

Lemma 7.8. *Let*

$$\sigma(t) = \sigma_{c,r}(t) := \begin{cases} (r \tanh t + c, r \operatorname{sech} t) & (0 < r < \infty), \\ (c, e^t) & (r = +\infty), \end{cases}$$

where $c \in \mathbb{R}$. *Then*

$$\pi^{-1} \circ \sigma(t) = (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}$$

for some $\boldsymbol{x} \in H^2$ *and* $\boldsymbol{v} \in U_{\boldsymbol{x}}H^2$.

Proof. By direct computation, the conclusion follows by setting

$$\boldsymbol{x} = \left(\frac{1 + c^2 + r^2}{2r}, \frac{-1 + c^2 + r^2}{2r}, \frac{c}{r}\right)^T, \qquad \boldsymbol{v} = (c, -c, 1)^T$$

when $0 < r < +\infty$. When $r = +\infty$, it is sufficient to set $\boldsymbol{x} = (1 + c^2/2, -c^2/2, c)^T$ and $\boldsymbol{v} = (-c^2/2, -1 + c^2/2, -c)^T$. □

Shortest path

Let $\boldsymbol{x} \in H^2$ and $\boldsymbol{v} \in U_{\boldsymbol{x}}H^2$, and set

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh t)\boldsymbol{x} + (\sinh t)\boldsymbol{v}. \tag{7.4}$$

Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1$, $\langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0$, and $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$, we have

Lemma 7.9. *The curve* $\gamma := \gamma_{\boldsymbol{x},\boldsymbol{v}}$ *in* (7.4) *is a curve on* H^2 *with* $\gamma(0) = \boldsymbol{x}$ *and* $\gamma'(0) = \boldsymbol{v}$. *Moreover,* t *is the arc-length parameter, that is,* $\langle \gamma'(t), \gamma'(t) \rangle = 1$.

Proposition 7.10. *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be two distinct points in* H^2 . *Then the shortest path joining* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is parametrized as* $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t)$, *where*

$$\boldsymbol{v} := \frac{\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{x}}{|\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{x}|}, \tag{7.5}$$

which is the arc in $\Pi_{\boldsymbol{x},\boldsymbol{y}} \cap H^2$, *where* $\Pi_{\boldsymbol{x},\boldsymbol{y}}$ *is the plane spanned by* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

Proof. First, the "straight line" on the upper-half plane $(\mathbb{R}^2_+, ds^2)$ is the shortest path. Let P and Q be two distinct points on $\mathbb{R}^2_+$. By a congruence as in Section 1, we may assume that $\mathrm{P} = (0, 1)$ and $\mathrm{Q} = (0, q)$ where $q > 0$. Then by the same argument as in Problem 5-2 in Section 5, the shortest path is the line segment on the y-axis. Since the corresponding path on H^2 is in the form $\gamma_{\boldsymbol{x},\boldsymbol{v}}$ by Lemma 7.8. Then by the same argument as Problem 6-2 in Section 6, we have the conclusion. □

Corollary 7.11. *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be two distinct points on* H^2. *Then the distance* $\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y})$ *of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is*

$$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^{-1}(-\langle \boldsymbol{x}, \boldsymbol{y} \rangle).$$

Proof. First, if we set $\boldsymbol{x} = (x_0, x_1, x_2)^T$ and $\boldsymbol{y} = (y_0, y_1, y_2)^T$,

$$\begin{aligned}\langle \boldsymbol{x}, \boldsymbol{y} \rangle &= -x_0y_0 + x_1y_1 + x_2y_2 = -\sqrt{1 + x_1^2 + x_2^2}\sqrt{1 + y_1^2 + y_2^2} + x_1y_1 + x_2y_2 \\ &\leqq -\sqrt{1 + x_1^2 + x_2^2}\sqrt{1 + y_1^2 + y_2^2} + \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} \leqq -1.\end{aligned}$$

Then $\cosh^{-1}(-\langle \boldsymbol{x}, \boldsymbol{y} \rangle)$ is a real number. Since shortest path joining $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t)$, where $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t_0) = \boldsymbol{y}$ for some $t_0 \in \mathbb{R}$:

$$\boldsymbol{y} = (\cosh t_0)\boldsymbol{x} + (\sinh t_0)\boldsymbol{v}.$$

Taking inner product with y, we have $-1 = \cosh(t_0)\boldsymbol{x}$. Here, since t is the arc-length parameter, $t_0 = \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y})$. Thus we have the conclusion. □

Example 7.12 (The hyperbolic Pythagorean theorem)**.** Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ be three non-co-linear points in H^2, and ξ, η, γ the straight lines joining $\boldsymbol{y}$ and $\boldsymbol{z}$, $\boldsymbol{x}$ and $\boldsymbol{z}$, and $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. If the angle between η and ζ at $\boldsymbol{x}$ is right-angle, then

$$\cosh X = \cosh Y \cosh Z, \qquad X = \operatorname{dist}(\boldsymbol{y}, \boldsymbol{z}), \quad Y = \operatorname{dist}(\boldsymbol{x}, \boldsymbol{z}), \quad Z = \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y})$$

holds.

In fact, represent the arcs η and ζ by $\gamma_{\boldsymbol{x},\boldsymbol{v}}$ and $\gamma_{\boldsymbol{x},\boldsymbol{w}}$, where

$$\boldsymbol{v} := \frac{\boldsymbol{z} + \langle \boldsymbol{x}, \boldsymbol{z} \rangle \boldsymbol{x}}{|\boldsymbol{z} + \langle \boldsymbol{x}, \boldsymbol{z} \rangle \boldsymbol{x}|}, \quad \boldsymbol{w} := \frac{\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{x}}{|\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{x}|}.$$

By assumption, $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, namely,

$$\langle \boldsymbol{z} + \langle \boldsymbol{x}, \boldsymbol{z} \rangle \boldsymbol{x}, \boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{x} \rangle = 0.$$

Hence $-\langle \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle \langle \boldsymbol{x}, \boldsymbol{z} \rangle$, and we have the conclusion by Corollary 7.11.

Various models of the hyperbolic plane

Poincaré disc model: Let

$$\text{(7.6)} \qquad \pi_P \colon H^2 \ni (x_0, x_1, x_2)^T \mapsto \frac{1 + x_0}{(}x_1, x_2) \ni D := \{(u, v) \in \mathbb{R}^2\,;\, u^2 + v^2 < 1\},$$

which is called the *stereographic projection.* The inverse of π_P is written as

$$\pi_P^{-1}(u, v) = \left(\frac{1 + u^2 + v^2}{1 - u^2 - v^2}, \frac{2u}{1 - u^2 - v^2}, \frac{2v}{1 - u^2 - v^2}\right).$$

The metric on D induced from H^2 is computed as

$$ds_P^2 := \frac{4}{(1 - u^2 - v^2)^2}(du^2 + dv^2).$$

The model (D, ds_P^2) is called the *Poincaré disc model* of the hyperbolic plane.

Klein model: Let

$$\text{(7.7)} \qquad \pi_C \colon H^2 \ni (x_0, x_1, x_2)^T \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \ni D := \{(\xi, \eta) \in \mathbb{R}^2\,;\, \xi^2 + \eta^2 < 1\},$$

which is called the *central projection.* The inverse and the induced metric ds_C^2 is represented as

$$\pi_C^{-1}(\xi, \eta) = \frac{1}{\sqrt{1 - \xi^2 - \eta^2}}(1, \xi, \eta)^T, \qquad ds_C^2 = \frac{1}{(1 - \xi^2 - \eta^2)}\big((1-\eta^2)\, d\xi^2 + 2\xi\eta\, d\xi\, d\eta + (1-\xi^2)\, d\eta^2\big).$$

The model (D, ds_C^2) is called the *Klein model* or the *projective model*, in which a "straight line" is a line segment of $D \subset \mathbb{R}^2$.